In this chapter we develop the theory which is the mainspring of the chromatic spectral sequence. Let \( K(n)_* = \mathbb{Z}/(p)[v_n, v_n^{-1}] \) have the \( BP_* \)-module structure obtained by sending all \( v_i \), with \( i \neq n \) to 0. Then define \( \Sigma(n) \) to be the Hopf algebra \( K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_* \). We will describe this explicitly as a \( K(n)_* \)-algebra below. Its relevance to the Adams–Novikov spectral sequence is the isomorphism \( (6.1.1) \)

\[
\text{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/I_n) \cong \text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*),
\]

which is input needed for the chromatic spectral sequence machinery described in Section 5.1. In combination with 6.2.4, this is the result promised in 1.4.9. Since \( \Sigma(n) \) is much smaller than \( BP_*(BP) \), this result is a great computational aid. We will prove it along with some generalizations in Section 1, following Miller and Ravenel [5] and Morava [2].

In Section 2 we study \( \Sigma(n) \), the \( n \)th Morava stabilizer algebra. We will show (6.2.5) that it is closely related to the \( \mathbb{Z}/(p) \)-group algebra of a pro-\( p \)-group \( S_n \) (6.2.3 and 6.2.4). \( S_n \) is the strict automorphism group \([i.e., the group of automorphisms \( f(x) \) having leading term \( x \)] of the height \( n \) formal group law \( F_n \) (see A2.2.17 for a description of the corresponding endomorphism ring). We use general theorems from the cohomology of profinite groups to show \( S_n \) is either \( p \)-periodic (if \( (p-1) | n \)) or has cohomological dimension \( n^2 \) (6.2.10).

In Section 3 we study this cohomology in more detail. The filtration of 4.3.24 leads to a May spectral sequence studied in 6.3.3 and 6.3.4. Then we compute \( H^1 \) (6.3.12) and \( H^2 \) (6.3.14) for all \( n \) and \( p \). The section concludes with computations of the full cohomology for \( n = 1 \) (6.3.21), \( n = 2 \) and \( p > 3 \) (6.3.22), \( n = 2 \) and \( p = 3 \) (6.3.24), \( n = 2 \) and \( p = 2 \) (6.3.27), and \( n = 3 \), \( p > 3 \) (6.3.32).

The last two sections concern applications of this theory. In Section 4 we consider certain elements \( \beta_{p'/p} \) in \( \text{Ext}^2(BP_*) \) for \( p > 2 \) analogous to the Kervaire invariant elements \( \beta_{2/2} \) for \( p = 2 \). We show (6.4.1) that these elements are not permanent cycles in the Adams–Novikov spectral sequence. A crucial step in the proof uses the fact that \( S_{p-1} \) has a subgroup of order \( p \) to detect a lot of elements in Ext. Theorem 6.4.1 provides a test that must be passed by any program to prove the Kervaire invariant conjecture: it must not generalize to odd primes!

In Section 5 we construct ring spectra \( T(m) \) satisfying \( BP_*(T(m)) = BP_*[t_1, \ldots, t_m] \) as comodules. The algebraic properties of these spectra will be exploited in the next chapter. We will show (6.5.5, 6.5.6, 6.5.11, and 6.5.12) that its Adams–Novikov \( E_2 \)-term has nice properties.

1. The Change-of-Rings Isomorphism

Our first objective is to prove
6.1.1. Theorem (Miller and Ravenel [5]). Let $M$ be a $BP_\ast (BP)$-comodule annihilated by $I_n = (p,v_1,\ldots,v_{n-1})$, and let $\overline{M} = M \otimes_{BP_\ast} K(n)_\ast$. Then there is a natural isomorphism

$$\operatorname{Ext}_{BP_\ast (BP)}(BP_\ast, v_n^{-1}M) = \operatorname{Ext}_{\Sigma(n)}(K(n)_\ast, \overline{M}).$$

Here $v_n^{-1}M$ denotes $v_n^{-1}BP_\ast \otimes_{BP_\ast} M$, which is a comodule (even though $v_n^{-1}BP_\ast$ is not) by 5.1.6.

This result can be generalized in two ways. Let

$$E(n)_\ast = v_n^{-1}BP_\ast / (v_{n+i} : i > 0)$$

and

$$E(n)_\ast (E(n)) = E(n)_\ast \otimes_{BP_\ast} BP_\ast (BP) \otimes_{BP_\ast} E(n)_\ast.$$  

It can be shown, using the exact functor theorem of Landweber [3], that $E(n)_\ast \otimes_{BP_\ast} BP_\ast (X)$ is a homology theory on $X$ represented by a spectrum $E(n)$ with $\pi_\ast (E(n)) = E(n)_\ast$, and with $E(n)_\ast (E(n))$ being the object defined above.

We can generalize 6.1.1 by replacing $\Sigma(n)$ with $E(n)_\ast (E(n))$ and relaxing the hypothesis on $M$ to the condition that it be $I_n$-nil, i.e., that each element (but not necessarily the entire comodule) be annihilated by some power of $I_n$. For example, $N^n$ of Section 5.1 is $I_n$-nil. Then we have

6.1.2. Theorem (Miller and Ravenel [5]). Let $M$ be $I_n$-nil and let $\overline{M} = M \otimes_{BP_\ast} E(n)_\ast$. Then there is a natural isomorphism

$$\operatorname{Ext}_{BP_\ast (BP)}(BP_\ast, v_n^{-1}M) = \operatorname{Ext}_{E(n)_\ast (E(n))}(E(n)_\ast, \overline{M}).$$

There is another variation due to Morava [2].\rec{10}{Regard $BP_\ast$ as a $\mathbb{Z}/2(p^n - 1)$-graded object and consider the homomorphism $\theta : BP_\ast \to \mathbb{Z}/(p)$ given by $\theta(v_n) = 1$ and $\theta(v_i) = 0$ for $i \neq n$. Let $I \subset BP_\ast$ be $\ker \theta$ and let $V_\theta$ and $VT_\theta$ denote the $I$-adic completions of $BP_\ast$ and $BP_\ast (BP)$. Let $E_\theta = V_\theta (v_{n+i} : i > 0)$ and $EH_\theta = E_\theta \otimes_{V_\theta} VT_\theta \otimes_{V_\theta} E_\theta$.

6.1.3. Theorem (Morava [2]). With notation as above there is a natural isomorphism

$$\operatorname{Ext}_{VT_\theta}(V_\theta, M) \cong \operatorname{Ext}_{EH_\theta}(E_\theta, \overline{M})$$

where $M$ is a $VT_\theta$-comodule and $\overline{M} = M \otimes_{V_\theta} E_\theta$. \hfill \box

Of these three results only 6.1.1 is relevant to our purposes so we will not prove the others in detail. However, Morava’s proof is more illuminating than that of Miller and Ravenel [5] so we will sketch it first.

Morava’s argument rests on careful analysis of the functors represented by the Hopf algebroids $VT_\theta$ and $EH_\theta$. First we need some general nonsense.

Recall that a groupoid is a small category in which every morphism is invertible. Recall that a Hopf algebroid $(A, \Gamma)$ over $K$ is a cogroupoid object in the category of commutative $K$-algebras; i.e., it represents a covariant groupoid-valued functor. Let $\alpha, \beta : G \to H$ be maps (functors) from the groupoid $G$ to the groupoid $H$. Since $G$ is a category it has a set of objects, $\text{Ob}(G)$, and a set of morphisms, $\text{Mor}(G)$, and similarly for $H$.\rec{9}
6.1.4. Definition. The functors $\alpha, \beta: G \to H$ are equivalent if there is a map $\theta: \text{Ob}(G) \to \text{Mor}(H)$ such that for any morphism $g: g_1 \to g_2$ in $G$ the diagram

$$
\begin{array}{ccc}
\alpha(g_1) & \xrightarrow{\alpha(g)} & \alpha(g_2) \\
\downarrow{\theta(g_1)} & & \downarrow{\theta(g_2)} \\
\beta(g_1) & \xrightarrow{\beta(g)} & \beta(g_2)
\end{array}
$$

commutes. Two maps of Hopf algebroids $a, b: (A, \Gamma) \to (B, \Sigma)$ are naturally equivalent if the corresponding natural transformations of groupoid-valued functors are naturally equivalent in the above sense. Two Hopf algebroids $(A, \Gamma)$ and $(B, \Sigma)$ are equivalent if there are maps $f: (A, \Gamma) \to (B, \Sigma)$ and $h: (B, \Sigma) \to (A, \Gamma)$ such that $hf$ and $fh$ are naturally equivalent to the appropriate identity maps.

Now we will show that a Hopf algebroid equivalence induces an isomorphism of certain Ext groups. Given a map $f: (A, \Gamma) \to (B, \Sigma)$ and a left $\Gamma$-comodule $M$, define a $\Sigma$-comodule $f^*(M)$ to be $B \otimes_A M$ with coactions

$$B \otimes_A M \to B \otimes_A \Gamma \otimes_A M \to B \otimes_B \Sigma \otimes_A M = \Sigma \otimes_B B \otimes_A M.$$  

6.1.5. Lemma. Let $f: (A, \Gamma) \to (B, \Sigma)$ a Hopf algebroid equivalence. Then there is a natural isomorphism $\text{Ext}_\Gamma(A, M) \cong \text{Ext}_\Sigma(B, f^*(M))$ for any $\Gamma$-comodule $M$.

Proof. It suffices to show that equivalent maps induce the same homomorphisms of Ext groups. An equivalence between the maps $a, b: (A, \Gamma) \to (B, \Sigma)$ is a homomorphism $\phi: \Gamma \to B$ with suitable properties, including $\phi_\eta_R = a$ and $\phi_\eta_L = b$. Since $\eta_R$ and $\eta_L$ are related by the conjugation in $\Gamma$, it follows that the two $A$-module structures on $B$ are isomorphic and that $a^*(M)$ is naturally isomorphic to $b^*(M)$. We denote them interchangeably by $M'$. The maps $a$ and $b$ induce maps of cobar complexes (A1.2.11) $C_\Gamma(M) \to C_\Sigma(M')$. A tedious routine verification shows that $\phi$ induces the required chain homotopy.

Now we consider the functors represented by $VT_\theta$ and $EH_\theta$. Recall that an Artin local ring is a commutative ring with a single maximal ideal satisfying the descending chain condition, i.e., the maximal ideal is nilpotent. If $A$ is such a ring with finite residue field $k$ then it is $W(k)$-module, where $W(k)$ is the Witt ring of Artin local rings. Let $\text{Art}_\theta$ denote the category of $\mathbb{Z}/(2(p^n - 1))$-graded Artin local rings whose residue field is an $F_q$-algebra. Now let $m_\theta = \ker \theta \subset B_\theta$. Then $B_\theta/m_\theta$ with the cyclic grading is is object in $\text{Art}_\theta$, so $V_\theta = \lim B_\theta/m_\theta$ is an inverse limit of such objects as is $VT_\theta$. For any $A \in \text{Art}_\theta$, we can consider $\text{Hom}^\omega(VT_\theta, A)$, the set of continuous ring homomorphisms from $VT_\theta$ to $A$. It is a groupoid-valued functor on $\text{Art}_\theta$ pro-represented by $VT_\theta$. (We have to say “pro-represented” rather than “represented” because $VT_\theta$ is not in $\text{Art}_\theta$.)

6.1.6. Proposition. $VT_\theta$ pro-represents the functor $\text{lifts}_\theta$ from $\text{Art}_\theta$ to the category of groupoids, defined as follows. Let $A \in \text{Art}_\theta$ have residue field $k$. The objects in $\text{lifts}_\theta(A)$ are $p$-typical liftings to $A$ of the formal group law over $k$ induced by the composite $B_\theta \xrightarrow{\theta} F_q \to k$, and morphisms in $\text{lifts}_\theta(A)$ are strict isomorphisms between such liftings.

6.1.7. Definition. Let $m_A \subset A$ be the maximal ideal for $A \in \text{Art}_\theta$. Given a homomorphism $f: F \to G$ of formal group laws over $A$, let $\bar{f}: \overline{G} \to \overline{G}$ denote their reductions mod $m_A$. $f$ is a $*$-isomorphism if $\bar{f}(x) = x$. 
6.1.8. Lemma. Let $F$ and $G$ be objects in $\text{lifts}_\theta(A)$. Then the map $\text{Hom}(F,G) \to \text{Hom}(F,G)$ is injective.

Proof. Suppose $f = 0$, i.e., $f(x) = 0 \mod m_A$. We will show that $f(x) \equiv 0 \mod m_A^r$ implies $f(x) \equiv 0 \mod m_A^{r+1}$ for any $r > 0$, so $f(x) = 0$ since $m_A$ is nilpotent. We have

$$G(f(x), f(y)) \equiv f(x) + f(y) \mod m_A^{2r}$$

since

$$G(x, y) \equiv x + y \mod (x, y)^2.$$ Consequently,

$$[p]_G(f(x)) \equiv pf(x) \mod m_A^{2r} \equiv 0 \mod m_A^{r+1}$$

since $p \in m_A$. On the other hand

$$[p]_G(f(x)) = f([p]_F(x))$$

and we know $[p]_F(x) \equiv x^p \mod m_A$ by A2.2.4. Hence $f([p]_F(x)) \equiv 0 \mod m_A^{r+1}$ gives the desired congruence $f(x) \equiv 0 \mod m_A^{r+1}$. \hfill \Box

Now suppose $f_1, f_2 : F \to G$ are $*$-isomorphisms (6.1.7) as is $g : G \to F$. Then $gf_1 = g f_2$ by 6.1.8 so $f_1 = f_2$; i.e. $*$-isomorphisms are unique. Hence we can make

6.1.9. Definition. $\text{lifts}_\theta^*(A)$ is the groupoid of $*$-isomorphism classes of objects in $\text{lifts}_\theta(A)$.

6.1.10. Lemma. The functors $\text{lifts}_\theta$ and $\text{lifts}_\theta^*$ are naturally equivalent.

Proof. There is an obvious natural transformation $\alpha : \text{lifts}_\theta \to \text{lifts}_\theta^*$, and we need to define $\beta : \text{lifts}_\theta^* \to \text{lifts}_\theta$, of each $*$-isomorphism class. Having done this, $\alpha \beta$ will be the identity on $\text{lifts}_\theta^*$ and we will have to prove $\beta \alpha$ is equivalent (6.1.4) to the identity on $\text{lifts}_\theta$.

The construction of $\beta$ is essentially due to Lubin and Tate [3]. Suppose $G_1 \in \text{lifts}_\theta(A)$ is induced by $\theta_1 : BP_\ast \to A$. Using A2.1.26 and A2.2.5 one can show that there is a unique $G_2 \in \text{lifts}_\theta(A)$ $*$-isomorphic to $G_1$ and induced by $\theta_2$ satisfying $\theta(v_{n+i}) = 0$ for all $i > 0$. We leave the details to the interested reader. As remarked above, the $*$-isomorphism from $G_1$ to $G_2$ is unique. The verification that $\beta \alpha$ is equivalent to the identity is straightforward. \hfill \Box

To prove 6.1.3, it follows from 6.1.5 and 6.1.10 that it suffices to show $E \text{H}_\theta$ pro-represents $\text{lifts}_\theta^*$. In the proof of 6.1.10 it was claimed that any suitable formal group law over $A$ is canonically $*$-isomorphic to one induced by $\theta : BP_\ast \to A$ which is such that it factors through $E_\theta$. In the same way it is clear that the morphism set of $\text{lifts}_\theta^*(A)$ is represented by $E \text{H}_\theta$, so 6.1.3 follows.

Now we turn to the proof of 6.1.1. We have a map $BP_\ast(BP) \to \Sigma(n)$ and we need to show that it satisfies the hypotheses of the general change-of-rings isomorphism theorem A1.3.12, i.e., of A1.1.19. These conditions are

\begin{enumerate}
\item[(6.1.11)]\label{6.1.11}
\begin{enumerate}
\item[(i)] the map $\Gamma' = BP_\ast(BP) \otimes_{BP_\ast} K(n)_* \to \Sigma(n)$ is onto
\item[(ii)] $\Gamma' \square_{\Sigma(n)} K(n)_*$ is a $K(n)_*$-summand of $\Gamma'$.\end{enumerate}
\end{enumerate}

Part (i) follows immediately from the definition $\Sigma(n) = K(n)_* \otimes_{BP_\ast} \Gamma'$. Part (ii) is more difficult. We prefer to replace it with its conjugate,
(ii) $K(n)_\ast \boxtimes_{\Sigma(n)_\ast} K(n)_\ast(BP)$ is a $K(n)_\ast$ summand of $K(n)_\ast BP$ which is defined to be $K(n)_\ast \otimes_{BP} BP_\ast(BP)$. Let $B(n)_\ast$ denote $v_n^{-1}BP_n/I_n$. Then the right $BP_\ast$-module structure on $K(n)_\ast(BP)$ induces a right $B(n)_\ast$-module structure.

6.1.12. LEMMA. There is a map

$$K(n)_\ast BP \to \Sigma(n) \otimes_{K(n)_\ast} B(n)_\ast,$$

which is an isomorphism of $\Sigma(n)$-comodules and of $B(n)_\ast$-modules, and which carries $1$ to $1$.

PROOF. Our proof is a counting argument, and in order to meet requirements of connectivity and finiteness, we pass to suitable “valuation rings”. Thus let

$$k(0)_\ast = \mathbb{Z}(p) \subset K(0)_\ast,$$

$$k(n)_\ast = F_p[v_n] \subset K(n)_\ast, \quad n > 0,$$

$$k(n)_\ast BP = k(n)_\ast \otimes_{BP} BP_\ast(BP) \subset K(n)_\ast BP,$$

$$b(n)_\ast = k(n)_\ast[u_1, u_2, \ldots] \subset B(n)_\ast,$$

where $u_k = v_n^{-1}v_{n+k}$.

It follows from A2.2.5 that in $k(n)_\ast BP$,

$$\eta_R(v_{n+k}) \equiv v_n t_k^p v_n^{-k} \pmod{(\eta_R(v_{n+1}), \ldots, \eta_R(v_{n+k-1}))}.$$

Hence $\eta_R : B(n)_\ast \to k(n)_\ast BP$ factors through an algebra map $b(n)_\ast \to k(n)_\ast BP$. It is clear from 6.1.13 that as a right $b(n)_\ast$-module, $k(n)_\ast BP$ is free on generators $t^\alpha = t_1^{\alpha_1}t_2^{\alpha_2} \ldots$ where $0 \leq \alpha_i < p^k$ and all but finitely many $\alpha_i$ are 0; in particular, it is of finite type over $b(n)_\ast$.

Now define

$$\sigma(n) = k(n)_\ast BP \otimes_{b(n)_\ast} k(n)_\ast \subset \Sigma(n);$$

by the above remarks $\sigma(n) = k(n)_\ast[t_1, t_2, \ldots]/(t_k^p - v_n^{k-1}t_k : k \geq 1)$ as an algebra. $(k(n)_\ast, \sigma(n))$ is clearly a sub-Hopf algebroid of $(K(n)_\ast, \Sigma(n))$, so $\sigma(n)$ is a Hopf algebra over the principal ideal domain $k(n)_\ast$.

The natural map $BP_\ast(BP) \to \sigma(n)$ makes $BP_\ast(BP)$ a left $\sigma(n)$-comodule, and this induces a left $\sigma(n)$-comodule structure on $k(n)_\ast BP$. We will show that the latter is an extended left $\sigma(n)$-comodule.

Define a $b(n)_\ast$-linear map $f : k(n)_\ast BP \to b(n)_\ast$ by

$$f(t^\alpha) = \begin{cases} 1 & \text{if } \alpha = (0, 0, \ldots) \\ 0 & \text{otherwise.} \end{cases}$$

Then $f$ satisfies the equations

$$f\eta_R = id : b(n)_\ast \to b(n)_\ast,$$

$$f \otimes_{b(n)_\ast} k(n)_\ast = \varepsilon : \sigma(n) \to k(n)_\ast.$$

Now let $\tilde{f}$ be the $\sigma(n)$-comodule map lifting $f$:

\[ \begin{array}{ccc}
  k(n)_\ast BP & \xrightarrow{\psi} & \sigma(n) \otimes_{k(n)_\ast} k(n)_\ast BP \\
  f & \downarrow & \sigma(n) \otimes f \\
  & \sigma(n) \otimes_{k(n)_\ast} b(n)_\ast & 
\end{array} \]
Since $\psi \bar{\eta}_R(x) = 1 \otimes \bar{\eta}_R(x)$, $\psi$ is $b(n)_*$-linear, so $\tilde{f}$ is too. We claim $\tilde{f}$ is an isomorphism. Since both sides are free of finite type over $b(n)_*$ it suffices to prove that $\tilde{f} \otimes_{b(n)_*} k(n)_*$ is an isomorphism. But 6.1.14 is then reduced to

$$\sigma(n) \xrightarrow{\Delta} \sigma(n) \otimes_{k(n)_*} \sigma(n)$$

$$\tilde{f} \otimes_{b(n)_*} k(n)_*$$

$$\sigma(n) \otimes_{k(n)_*} k(n)_*$$

so the claim follows from unitarity of $\Delta$.

Now the map $K(n)_* \otimes_{k(n)_*} \tilde{f}$ satisfies the requirements of the lemma. □

6.1.15. Corollary. $\bar{\eta}_R : B(n)_* \rightarrow K(n)_* \boxtimes_{\Sigma(n)} K(n)_* BP$ is an isomorphism of $B(n)_*$-modules.

Proof. The natural isomorphism

$$B(n)_* \rightarrow K(n)_* \boxtimes_{\Sigma(n)} (\Sigma(n) \otimes_{K(n)_*} B(n)_*)$$

is $B(n)_*$-linear and carries 1 to 1. Hence

$$K(n)_* \boxtimes_{\Sigma(n)} (\Sigma(n) \otimes_{K(n)_*} B(n)_*) \cong B(n)_* \rightarrow K(n)_* \boxtimes_{\Sigma(n)} K(n)_* BP$$

commutes, and $\bar{\eta}_R$ is an isomorphism. □

Hence 6.1.11(ii) follows from the fact that $K(n)_*$ is a summand of $\Sigma(n)$, and 6.1.1 is proved. From the proof of 6.1.12 we get an explicit description of $\Sigma(n)$, namely

6.1.16. Corollary. As an algebra

$$\Sigma(n) = K(n)_*[t_1, t_2, \ldots]/(v_n t_i^{p^n} - v_n^{p^i} t_i: i > 0).$$

Its coproduct is inherited from $BP_*(BP)$, i.e., a suitable reduction of 4.3.13 holds.

2. The Structure of $\Sigma(n)$

To study $\Sigma(n)$ it is convenient to pass to the corresponding object graded over $\mathbb{Z}/2(p^n - 1)$. Make $F_p$ a $K(n)_*$-module by sending $v_n$ to 1, and let $S(n) = \Sigma(n) \otimes_{K(n)_*} F_p$. For a $\Sigma(n)$-comodule $M$ let $\overline{M} = M \otimes_{K(n)_*} F_p$, which is easily seen to be an $S(n)$-comodule. The categories of $\Sigma(n)$- and $S(n)$-comodules are equivalent and we have

6.2.1. Proposition. For a $\Sigma(n)$-comodule $M$,

$$\operatorname{Ext}_{\Sigma(n)}(K(n)_*, M) \otimes_{K(n)_*} F_p \cong \operatorname{Ext}_{S(n)}(F_p, \overline{M}).$$
2. The Structure of \( \Sigma(n) \)

We will see below (6.2.5) that if we regard \( S(n) \) and \( \overline{M} \) as graded merely over \( \mathbb{Z}/(2) \), there is a way to recover the grading over \( \mathbb{Z}/(p^n - 1) \). If \( M \) is concentrated in even dimensions (which it is in most applications) then we can regard \( \overline{M} \) and \( S(n) \) as ungraded objects. Our first major result is that \( S(n) \otimes \mathbb{F}_{p^n} \) (ungraded) is the continuous linear dual of the \( \mathbb{F}_{p^n} \)-group algebra of a certain profinite group \( S_n \) to be defined presently.

6.2.2. Definition. The topological linear dual \( S(n)^* \) of \( S(n) \) is as follows. [In Ravenel [5] \( S(n)^* \) and \( S(n) \) are denoted by \( S(n) \) and \( S(n)_* \), respectively.] Let \( S(n)_{(i)} \) be the sub-Hopf algebra of \( S(n) \) generated by \( \{ t, \ldots, t_i \} \). It is a vector space of rank \( p^{ni} \) and \( S(n) = \lim \hom(S(n)_{(i)}, \mathbb{F}_p) \), equipped with the inverse limit topology. The product and coproduct in \( S(n) \) give maps of \( S(n)^* \) to and from the completed tensor product

\[
S(n)^* \hat{\otimes} S(n)^* = \lim \hom(S(n)_{(i)} \otimes S(n)_{(j)}, \mathbb{F}_p).
\]

To define the group \( S_n \) recall the \( \mathbb{Z}_p \)-algebra \( E_n \) of A2.2.16, the endomorphism ring of a height \( n \) formal group law. It is a free \( \mathbb{Z}_p \)-algebra of rank \( n^2 \) generated by \( \omega \) and \( S \), where \( \omega \) is a primitive \( (p^n - 1) \)th root of units, \( S = \omega^p S \), and \( S^n = p \). \( S_n \subset E_n^* \), is the group of units congruent to 1 mod \( (S) \), the maximal ideal in \( E_n \). \( S_n \) is a profinite group, so its group algebra \( \mathbb{F}_{p^n}[S_n] \) has a topology and is a profinite Hopf algebra. \( S_n \) is also a \( p \)-adic Lie group; such groups are studied by Lazard [4].

6.2.3. Theorem. \( S(n)^* \otimes \mathbb{F}_q \equiv \mathbb{F}_q[S_n] \) as profinite Hopf algebras, where \( q = p^n \), \( S_n \) is as above, and we disregard the grading on \( S(n)^* \).

Proof. First we will show \( S(n)^* \otimes \mathbb{F}_q \), is a group algebra. According to Sweedler [1], Proposition 3.2.1, a cocommutative Hopf algebra is a group algebra if it has a basis of group-like elements, i.e., of elements \( x \) satisfying \( \Delta x = x \otimes x \). This is equivalent to the existence of a dual basis of idempotent elements \( \{ y \} \) satisfying \( y_i^2 = y_i \), and \( y_i y_j = 0 \) for \( i \neq j \). Since \( S(n) \otimes \mathbb{F}_q \), is a tensor product of algebras of the form \( R = \mathbb{F}_q[t]/(t^q - t) \), it suffices to find such a basis for \( R \). Let \( a \in \mathbb{F}_q^\times \) be a generator and let

\[
r_i = \begin{cases} 
- \sum_{0<j<q} (a^i t)^j & \text{for } 0 < i < q, \\
1 - t^{q-1} & \text{for } i = 0.
\end{cases}
\]

Then \( \{ r_i \} \) is such a basis, so \( S(n)^* \otimes \mathbb{F}_q \), is a group algebra.

Note that tensoring with \( \mathbb{F}_q \) cannot be avoided, as the basis of \( R \) is not defined over \( \mathbb{F}_p \).

For the moment let \( G_n \) denote the group satisfying \( \mathbb{F}_p[G_n] \cong S(n)^* \otimes \mathbb{F}_q \). To get at it we define a completed left \( S(n) \)-comodule structure on \( \mathbb{F}_q[[x]] \), thereby defining a left \( G_n \)-action. Then we will show that it coincides with the action of \( S_n \) as formal group law automorphisms given by A2.2.17.

We now define the comodule structure map

\[
\psi: \mathbb{F}_q[[x]] \to S(n) \otimes \mathbb{F}_q[[x]]
\]

to be an algebra homomorphism given by

\[
\psi(x) = \sum_{i \geq 0} t_i \otimes x^{p^i},
\]
where \( t_0 = 1 \) as usual. To verify that this makes sense we must show that the following diagram commutes.

\[
\begin{array}{ccc}
F_q[[x]] & \xrightarrow{\psi} & S(n) \otimes F_q[[x]] \\
\downarrow \psi & & \downarrow (\Delta \otimes 1) \\
S(n) \otimes F_q[[x]] & \xrightarrow{1 \otimes \psi} & S(n) \otimes S(n) \otimes F_q[[x]] \\
\end{array}
\]

for which we have

\[
(\Delta \otimes 1)\psi(x) = (\Delta \otimes 1) \sum_{i \geq 0} t_i \otimes x^{p^i}
= \sum_{i \geq 0} \left( \sum_{j-k=i} t_j \otimes t_k^p \right) \otimes x^{p^i}
= \sum_{j-k \geq 0} t_j \otimes t_k^p \otimes x^{p^{i+k}}
\]

This can be seen by inserting \( x \) as a dummy variable in 4.3.12. We also have

\[
(1 \otimes \psi)\psi(x) = (1 \otimes \psi) \left( \sum_{i \geq 0} t_i \otimes x^{p^i} \right)
= \sum_{j \geq 0} \left( \sum_{i \geq 0} t_j \otimes x^{p^i} \right)^{p^i}
= \sum_{i,j \geq 0} t_i \otimes t_j^p \otimes x^{p^{i+j}}.
\]

The last equality follows from the fact that \( F(x^p, y^p) = F(x, y)^p \). The linearity of \( \psi \) follows from A2.2.20(b), so \( \psi \) defines an \( S(n) \otimes F_q \)-comodule structure on \( F_q[[x]] \).

We can regard the \( t_i \), as continuous \( F_q \)-valued functions on \( G_n \) and define an action of \( G_n \) on the algebra \( F_q[[x]] \) by

\[
g(x) = \sum_{i \geq 0} t_i(g)x^{p^i}
\]

for \( g \in G_n \). Hence \( G(x) = x \) iff \( g = 1 \), so our representation is faithful.

We can embed \( G_n \) in the set of all power series of the form \( \sum_{i \geq 0} a_i x^{p^i} \) which is \( E_n \) by A2.2.20 so the result follows.

6.2.4. Corollary. If \( M \) is an ungraded \( S(n) \)-comodule, then 6.2.3 gives a continuous \( S_n \)-action on \( M \otimes F_q \), and

\[
\text{Ext}^*_S(\mathbf{F}_q, M) \otimes \mathbf{F}_q = H_c^*(G_n, M \otimes F_q)
\]

where \( H_c^* \) denotes continuous group cohomology.

To recover the grading on \( S(n) \otimes M \), we have an action of the cyclic group of order \( q-1 \) generated by \( \bar{\omega}^i \omega^i \) via conjugation in \( E_n \).

6.2.5. Proposition. The eigenspace of \( S(n) \otimes \mathbf{F}_q \) with eigenvalue \( \bar{\omega}^i \) is the component \( S(n)_{2i} \otimes \mathbf{F}_q \) of degree \( 2i \).  \( \square \)
2. THE STRUCTURE OF $\Sigma(n)$

Proof. The eigenspace decomposition is multiplicative in the sense that if $x$ and $y$ are in the eigenspaces with eigenvalues $\bar{\omega}^i$ and $\bar{\omega}^j$, respectively, the $xy$ is in the eigenspace with eigenvalue $\bar{\omega}^{i+j}$. Hence it suffices to show that $t_k$ is in the eigenspace with eigenvalue $\bar{\omega}^{p^k-1}$.

To see this we compute the conjugation of $t_kS^k \in E_n$ by $\omega$ and we have $\omega^{-1}(t_kS^k)\omega = \omega^{-1}t_k\omega^{p^k}S^k = \omega^{p^k-1}t_kS^k$.

Corollary 6.2.4 enables us to apply certain results from group cohomology theory to our situation. First we give a matrix representation of $E_n$ over $W(F_q)$.

6.2.6. Proposition. Let $e = \sum_{0 \leq i \leq n} e_iS^i$ with $e_i \in W(F_q)$ be an element of $E_n$. Define an $n \times n$ matrix $(e_{i,j})$ over $W(F_q)$ by

$$e_{i+1,j+1} = \begin{cases} e_{j-i} & \text{for } i \leq j \\ pe_{j-i}^{n} & \text{for } i > j. \end{cases}$$

Then (a) this defines a faithful representation of $E_n$; (b) the determinant $|e_{i,j}|$ lies in $Z_p$.

Proof. Part (a) is straightforward. For (b) it suffices to check that $\omega$ and $S$ give determinants in $Z_p$.

We can now define homomorphisms $c: Z_p \to S_n$ and $d: S_n \to Z_p$ for $p > 2$, and $c: Z_2^\times \to S_n$ and $d: Z_2^\times$ for $p = 2$ by identifying $S_n$ with the appropriate matrix group. $(Z_p$ is to be regarded here as a subgroup of $Z_2^\times$.) Let $d$ be the determinant for all primes. For $p > 2$ let $e(x) = \exp(pxI)$, where $I$ is the $n \times n$ identity matrix and $x \in Z_p$; for $p = 2$ let $e(x) = xI$ for $x \in Z_2^\times$.

6.2.7. Theorem. Let $S^1_n = \ker d$.

(a) If $p > 2$ and $p \nmid n$ then $S_n \cong Z_p \oplus S^1_n$.

(b) If $p = 2$ and $n$ is odd then $S_n \cong S^1_n \oplus Z_2^\times$.

Proof. In both cases one sees that $\text{im } c$ lies in the center of $S_n$ (in fact $\text{im } c$ is the center of $S_n$) and is therefore a normal subgroup. The composition $dc$ is multiplication by $n$ which is an isomorphism for $p \nmid n$, so we have the desired splitting.

We now describe an analogous splitting for $S(n)$. Let $A^* = F_p[S_1]$, for $p > 2$ and $A^* = F_2[S_2^\times]$ for $p = 2$. Let $A_\sigma$ be the continuous linear dual of $A$.

6.2.8. Proposition. As an algebra $A = F_p[u_1, u_2, \ldots]/(u_1 - u_1^p)$. The coproduct $\Delta$ is given by

$$\sum_{i \geq 0} G \Delta(u_i) = \sum_{i,j \geq 0} G u_i \otimes u_j$$

where $u_0 = 1$ and $G$ is the formal group law with

$$\log_G(X) = \sum \frac{x^{p^i}}{p^i}.$$

Proof. Since $A \cong F_p[S_1]$, this follows immediately from 6.2.3.

We can define Hopf algebra homomorphisms $c_\sigma: S(n) \otimes F_q \to A \otimes F_q$ and $d_\sigma: A \otimes F_q \to S(n) \otimes F_q$ dual to the group homomorphisms $c$ and $d$ defined above.
6.2.9. Theorem. There exist maps $c_*: S(n) \to A$ and $d_*: A \to S(n)$ corresponding to those defined above, and for $p \nmid n$, $S(n) \cong A \otimes B$, where $B \otimes \mathbf{F}_p$, is the continuous linear dual of $\mathbf{F}_q[S_n^1]$, where $S_n^1$ is defined in 6.2.7.

Proof. We can define $c_*$ explicitly by

$$c_* t_i = \begin{cases} u_{i/n} & \text{if } n \mid i \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that this is a homomorphism corresponding to the $c_*$ defined above. In lieu of defining $d_*$ explicitly we observe that the determinant of $\sum_{i \geq 0} t_i S^i$, where $t_i \in W(F_q)$ and $t_i = t_i^1$, is a power series in $p$ whose coefficients are polynomials in the $t_i$ over $\mathbf{Z}_p$. It follows that $d_*$ can be defined over $\mathbf{F}_p$. The splitting then follows as in 6.2.7. \hfill \square

Our next result concerns the size of $\text{Ext}_{S(n)}(\mathbf{F}_p, \mathbf{F}_p)$, which we abbreviate by $H^*(S(n))$.

6.2.10. Theorem. 
(a) $H^*(S(n))$ is finitely generated as an algebra.
(b) If $(p - 1) \nmid n$, then $H^i(S(n)) = 0$ for $i > n^2$ and $H^i(S(n)) = H^{n^2-i}(S(n))$ for $0 \leq i \leq n^2$, i.e., $H^*(S(n))$ satisfies Poincaré duality.
(c) If $(p - 1) \mid n$, then $H^*(S(n))$ is $p$-periodic, i.e., there is some $x \in H^i(S(n))$ such that $H^*(S(n))$ above some finite dimension is a finitely generated free module over $\mathbf{F}_p[x]$. \hfill \square

We will prove 6.2.10(a) below as a consequence of the open subgroup theorem (6.3.6), which states that every sufficiently small open subgroup of $S_n$ has the same cohomology as $\mathbf{Z}_p^{n^2}$. Then (c) and the statement in (b) of finite cohomological dimension are equivalent to saying that the Krull dimension of $H^*(S(n))$ is 1 or 0, respectively. Recall that the Krull dimension of a Noetherian ring $R$ is the largest $d$ such that there is an ascending chain $p_0 \subset p_1 \subset \cdots \subset p_d$ of nonunit prime ideals in $R$. Roughly speaking, $d$ is the number of generators of the largest polynomial algebra contained in $R$. Thus $d = 0$ iff every element in $R$ is nilpotent, which in view of (a) implies (b). If $d = 1$ and $R$ is a graded $\mathbf{F}_p$-algebra, then every element in $R$ has a power in $\mathbf{F}_p[x]$ for a fixed $x \in R$. $R$ is a module over $\mathbf{F}_p[x]$, which is a principal ideal domain. Since $H^*(S(n))$ is graded and finitely generated, it is a direct sum of cyclic modules over $\mathbf{F}_p[x]$. More specifically it is a direct sum of a torsion module (where each element is annihilated by some power of $x$) and a free module. Since it is finitely generated, the torsion must be confined to low dimensions, and $H^*(S(n))$ is therefore a free $\mathbf{F}_p[x]$-module in high dimensions, so (a) implies (c).

The following result helps determine the Krull dimension.

6.2.11. Theorem (Quillen [3]). For a profinite group $G$ the Krull dimension of $H^*(G; \mathbf{F}_p)$ is the maximal rank of an elementary abelian $p$-subgroup of $G$, i.e., subgroup isomorphic to $(\mathbf{Z}/(p))^d$. \hfill \square

To determine the maximal elementary abelian subgroup of $S_n$, we use the fact that $D_n = E_n \otimes \mathbf{Q}$ is a division algebra over $\mathbf{Q}_p$ (A2.2.16), so if $G \subset S_n$ is abelian, then the $\mathbf{Q}_p$-vector space in $D_n$ spanned by the elements of $G$ is a subfield $K \subset D_n$. Hence the elements of $G$ are all roots of unity, $G$ is cyclic, and the Krull dimension is 0 or 1.
6.2.12. Theorem. A degree $m$ extension $K$ of $\mathbb{Q}_p$ embeds in $D_n$ iff $m \mid n$.

Proof. See Serre [1, p. 138] or Cassels and Fröhlich [1, p. 202], □

By 6.2.11 $H^*(S(n))$ has Krull dimension 1 iff $S_n$ contains $p$th roots of unity. Since the field $K$ obtained by adjoining such roots to $\mathbb{Q}_p$ has degree $p - 1$, 6.2.12 gives 6.2.10(c) and the finite cohomological dimension statement in (b). For the rest of (b) we rely on theorem V.2.5.8 of Lazard [4], which says that if $S_n$ (being an analytic pro-$p$-group of dimension $n^2$) has finite cohomological dimension, then that dimension is $n^2$ and Poincaré duality is satisfied.

The following result identifies some Hopf algebra quotients of $S(n) \otimes \mathbb{F}_p\mathbb{n}$. These are related to the graded Hopf algebras $\Sigma_A(n)$ discussed in Ravenel [10]. More precisely, $S(d, f)_a$ is a nongraded form of $\Sigma_A(d/f)$, where $A$ is the ring of integers in an extension $K$ (depending on $a$) of $\mathbb{Q}_p$ of degree $fn/d$ and residue degree $f$.

6.2.13. Theorem. Let $a \in \mathbb{F}_p$ be a $(p^n - 1)$th root of unity, let $d$ divide $n$, and let $f$ divide $d$. Then there is a Hopf algebra

$$S(d, f)_a = \mathbb{F}_p^n[t_f, t_{2f}, \ldots]/(t_i^{p^d} - a_i t_f : i > 0)$$

where $a_i = a^{p^d-1}$, and a surjective homomorphism

$$\theta: S(n) \otimes \mathbb{F}_p\mathbb{n} \rightarrow S(d, f)_a$$

given by

$$t_i \mapsto \begin{cases} t_i & \text{if } f \mid i \\ 0 & \text{otherwise.} \end{cases}$$

The coproduct on $S(d, f)_a$ is determined by the one on $S(n)$. This Hopf algebra is cocommutative when $f = d$.

Proof. We first show that the algebra structure on $S(d, f)_a$ is compatible with that on $S(n)$. The relation $t_i^{p^d} = a_i t_f$ implies

$$t_i^{p^d} = (a_i t_f)^{p^d} = a_i^{(p^d-1)/(p^d-1)} t_i t_f = a_i^{(p^d-1)/(p^d-1)} t_i t_f$$

$$t_i^{p^d} = a_i^{(p^d-1)/(p^d-1)} t_i t_f$$

$$\vdots$$

$$t_i^{p^n} = a_i^{(p^n-1)/(p^n-1)} t_i t_f = t_i t_f,$$

so $\theta$ exists as an algebra map.

For the coproduct in $S(n)$ we have

$$\sum_{i \geq 0} F \Delta(t_i) x^{p^i} = \sum_{i,j \geq 0} F t_i \otimes t_j^{p^j} x^{p^{i+j}}$$

(where $x$ is a dummy variable) which induces

$$\sum_{i \geq 0} F \Delta(t_i) x^{p^i} = \sum_{i,j \geq 0} F t_i \otimes t_j^{p^j} x^{p^{i+j}}$$

in $S(d, f)_a$. We need to show that this is compatible with the multiplicative relations. We can write $if = kd + \ell f$ with $0 \leq \ell f < d$, so we can rewrite the above
as
\[ \sum_{i \geq 0} F \Delta(t_{ij})x^{p^{i+j}} = \sum_{i, j \geq 0} t_{ij} \otimes t'_{ij} x^{p^{i+j}} \]
\[ = \sum_{i, j \geq 0} \alpha_{ij}(p^{d-1}/p^{a-1})t_{ij} \otimes t'_{ij} x^{p^{i+j}} \]
\[ = \sum_{i, j \geq 0} \alpha_{ij}(p^{d-1})(p^{d-1})t_{ij} \otimes t'_{ij} x^{p^{i+j}}, \]
which gives a well defined coproduct in \( S(d, f)_a \).

If \( f = d \) then the right hand side simplifies to
\[ \sum_{i, j \geq 0} \alpha_{ij}(p^{d-1})t_{ij} \otimes t'_{ij} x^{p^{i+j}}, \]
which is cocommutative as claimed.

\[ \square \]

3. The Cohomology of \( \Sigma(n) \)

In this section we will use a spectral sequence (A1.3.9) based on the filtration of \( \Sigma(n) \) induced by the one on \( BP_*(BP)/I_n \) given in 4.3.24. We have

6.3.1. THEOREM. Define integers \( d_{n,i} \) by
\[ d_{n,i} = \begin{cases} 0 & \text{if } i \leq 0 \\ \max(i, pd_{n,i-n}) & \text{for } i > 0. \end{cases} \]
Then there is a unique increasing filtration of the Hopf algebra \( S(n) \) with \( \deg t_{i,j} = d_{n,i} \) for \( 0 \leq j < n \). \[ \square \]

The following is a partial description of the coproduct in the associated graded object \( E^0 S(n) \). For large \( i \) we need only partial information about the coproduct on \( t_{i,j} \) in order to prove Theorem 6.3.3.

6.3.2. THEOREM. Let \( E^0 S(n) \) denote the associated bigraded Hopf algebra. Its algebra structure is
\[ E^0 S(n) = T(t_{i,j} : i > 0, \quad j \in \mathbb{Z}/(n)), \]
where \( T(\cdot) \) denotes the truncated polynomial algebra of height \( p \) on the indicated elements and \( t_{i,j} \) corresponds to \( t^{p^i}_{i,j} \). The coproduct is induced by the one given in 4.3.34. Explicitly, let \( m = pn/(p-1) \). Then
\[ \Delta(t_{i,j}) = \begin{cases} \sum_{0 \leq k \leq i} t_{k,j} \otimes t_{i-k,k+j} & \text{if } i < m, \\ \sum_{0 \leq k \leq i} t_{k,j} \otimes t_{i-k,k+j} + \tilde{b}_{i-n,j+n-1} & \text{if } i = m, \\ t_{i,j} \otimes 1 + 1 \otimes t_{i,j} + \tilde{b}_{i-n,j+n-1} & \text{mod}(t_{k,\ell} : k \leq i-n-1) & \text{if } i > m, \end{cases} \]
where \( t_{0,j} = 1 \) and \( \tilde{b}_{i,j} \) corresponds to the \( b_{i,j} \) of 4.3.14. \[ \square \]

As in the case of the Steenrod algebra, the dual object \( E_n S(n)^* \) is primitively generated and is the universal enveloping algebra of a restricted Lie algebra \( L(n) \). \( L(n) \) has basis \( \{ x_{i,j} : i > 0, \ j \in \mathbb{Z}/(n) \} \), where \( x_{i,j} \) is dual to \( t_{i,j} \).
6.3.3. **Theorem.** $E_0S(n)^*$ is the restricted enveloping algebra on primitives $x_{i,j}$ with bracket

$$[x_{i,j}, x_{k,l}] = \begin{cases} \delta_{i+k}^j x_{i+k,j} - \delta_{k+1}^j x_{i+k,t} & \text{for } i + k \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

where $m$ is the largest integer not exceeding $n/(p-1)$, and $\delta_s^t = 1$ iff $s \equiv t \mod (n)$ and $\delta_s^t = 0$ otherwise. The restriction $\xi$ is given by

$$\xi(x_{i,j}) = \begin{cases} x_{i+n,j+1} & \text{if } i > n/(p-1) \\ x_{2n,j} + x_{2n,j+1} & \text{if } i = n \text{ and } p > 2 \\ 0 & \text{if } i < n/p - 1. \end{cases}$$

The formula for the restriction was given incorrectly in the first edition, and this error led to an incorrect description in 6.3.24 of the multiplicative structure of $H^*(S(2))$ for $p = 3$. The correct description is due to Henn [1] and will be given below. The corrected restriction formula was given to me privately by Ethan Devinatz.

**Proof of 6.3.3.** The formula for the bracket can be derived from 6.3.2 as follows. The primitive $x_{i,j}$ is dual to $t_{i,j}$. The bracket has the form

$$[x_{i,j}, x_{k,l}] = \sum_{m,n} c_{i,j,k,l}^{a,b} x_{a,b},$$

where the coefficient $c_{i,j,k,l}^{a,b}$ is nonzero only if the coproduct expansion on $t_{a,b}$ contains a term of the form $t_{i,j} \otimes t_{k,l}$ or $t_{k,l} \otimes t_{i,j}$. This can happen only when the two expressions have the same bidegree. This means that

$$d_{n,a} = d_{n,i} + d_{n,k} \quad \text{and} \quad 2p^{i}(p^a - 1) \equiv 2p^{j}(p^i - 1) + 2p^{k}(p^j - 1) \mod 2(p^n - 1).$$

This happens only when $a = i + k \leq m$ and $b = j$ or $\ell$. Inspection of the coproduct formula leads to indicated Lie bracket.

The restriction requires more care. For finding the restriction on $x_{i,j}$ it suffices to work in the subalgebra of $E_0S(n)^*$ generated by $x_{k,\ell}$ for $k \geq i$.

It is also dual to passing to the quotient of $E^0S(n)$ obtained by killing $t_{k,\ell}$ for $k < i$. Hence description of $\Delta(t_{i,j})$ for $i > m$ given in 6.3.2 is sufficient for our purposes.

When $i > m$ we have

$$\Delta(t_{i,j}) = t_{i,j} \otimes 1 + 1 \otimes t_{i,j} + \delta_{i-n,j-1} \otimes t_{i-n,j-1} = t_{i,j} \otimes 1 + 1 \otimes t_{i,j} - \sum_{0<\ell<p} p^{-1} \binom{p}{\ell} t_{i-n,j-1}^{\ell} \otimes t_{i-n,j-1}^{p-\ell},$$

so for $i > n/(p-1)$,

$$\Delta(t_{i+n,j+1}) \equiv t_{i+n,j+1} \otimes 1 + 1 \otimes t_{i+n,j+1} - \sum_{0<\ell<p} p^{-1} \binom{p}{\ell} t_{i,j}^{\ell} \otimes t_{i,j}^{p-\ell} \mod (t_{k,\ell}: k \leq i-1).$$
For brevity let $B = E(S(n))/(p_{k,i}: k \leq i - 1)$ and let $\mathcal{E} = B/F_p$ denote the unit coideal, the dual of the augmentation ideal in $B^*$.

It follows that under the reduced iterated coproduct

$$B \xrightarrow{\Delta^{p-1}} B^p \xrightarrow{} \mathcal{E}^{\otimes p}$$

we have

$$t_{i+n,j+1} \mapsto t_{i,j} \otimes t_{i,j} \otimes \cdots \otimes t_{i,j},$$

which leads to the desired value of $\xi(x_{i,j})$ for $i > n/(p - 1)$. The argument for $i = n/p - 1$ and $p$ odd is similar.

For the case $p = 2$ and $i = n$, 6.3.2 gives

$$\Delta(t_{2n,j}) = \sum_{0 \leq k \leq 2n} t_{k,j} \otimes t_{2n-k,k+j} + \bar{b}_{n,j-1}$$

$$= t_{n,j-1} \otimes t_{n,j-1} + \sum_{0 \leq k \leq 2n} t_{k,j} \otimes t_{2n-k,k+j}$$

$$= t_{n,j-1} \otimes t_{n,j-1} + t_{n,j} \otimes t_{n,j}$$

$$+ \sum_{0 \leq k < n} (t_{k,j} \otimes t_{2n-k,j+k} + t_{2n-k,j} \otimes t_{k,j-k}),$$

and the formula for $\xi(x_{i,j})$ follows.

For $i < n/(p - 1)$ there are no terms in $\Delta(t_{i+n,k})$ for any $k$ that would lead to a nontrivial restriction on $x_{i,j}$. $\square$

Recall that Theorem 6.2.3 identifies $S(n)^* \otimes F_q$ with the group ring $F_q[S_n]$ and that $S_n$ is the group of units in the $\mathbb{Z}_p$-algebra $E_n$ congruent to 1 modulo the maximal ideal $(S)$. Killing the first few $t_i$s in $S(n)$ as we did in the proof above corresponds to replacing the group $S_n$ by the subgroup of units congruent to 1 modulo a power of $(S)$.

Let $L(n)$ be the Lie algebra without restriction with basis $x_{i,j}$ and bracket as above. We now recall the main results of May [2].

6.3.4. **Theorem.** There are spectral sequences

(a) $E_2 = H^*(L(n)) \otimes P(h_{i,j}) \Rightarrow H^*(E_0S(n)^*)$,

(b) $E_2 = H^*(E_0S(n)^*) \Rightarrow H^*(S(n))$,

where $h_{i,j} \in H^{2p^j}(E_0S(n)^*)$ with internal degree $2p^{j+1}(p^i - 1)$ and $P(\cdot)$ is the polynomial algebra on the indicated generators. $\square$

Now let $L(n, k)$ be the quotient of $L(n)$ obtained by setting $x_{i,j} = 0$ for $i > k$. Then our first result is

6.3.5. **Theorem.** The $E_2$-term of the first May spectral sequence [6.3.4(a)] may be replaced by $H^*(L(n, m)) \otimes P(h_{i,j}: i \leq m - n)$, where $m = \lfloor m/(p - 1) \rfloor$ as before.

**Proof.** By 6.3.3 $L(n)$ is the product of $L(n, m)$ and an abelian Lie algebra,

$$H^*(L(n)) \cong H^*(L(n, m)) \otimes E(h_{i,j}: i > m),$$

where $E(\cdot)$ denotes the exterior algebra on the indicated generators and $h_{i,j} \in H^1L(n)$ is the element corresponding to $x_{i,j}$. It also follows from 6.3.4 that the appropriate differential will send $h_{i,j}$ to $-b_{i-n,j-1}$ for $i > m$. It follows that the entire spectral sequence decomposes as a tensor product of two spectral sequences,
one with the $E_2$-term indicated in the statement of the theorem, and the other having $E_2 = E(h_{i,j}) \otimes P(h_{i-n,j})$ with $i > m$ and $E_\infty = F_p$. □

If $n < p - 1$ then 6.3.5 gives a spectral sequence whose $E_2$-term is $H^*(L(n, n))$, showing that $H^*(S(n))$ has cohomological dimension $n^2$ as claimed in 6.2.10(b).

In Ravenel [6] we claimed erroneously that the spectral sequence of 6.3.4(b) collapses for $n < p - 1$. The argument given there is incorrect. For example, we have reason to believe that for $p = 11$, $n = 9$ the element 

$$(h_{1,0}h_{2,0} \cdots h_{7,0})(h_{2,8}h_{3,7} \cdots h_{7,3})$$

supports a differential that hits a nonzero multiple of 

$$h_{1,0}h_{2,0}(h_{1,8}h_{2,7} \cdots h_{6,3})(h_{2,1}h_{3,1} \cdots h_{6,1}).$$

We know of no counterexample for smaller $n$ or $p$.

Now we will prove 6.2.10(a), i.e., that $H^*(S(n))$ is finitely generated as an algebra. For motivation, the following is a special case of a result in Lazard [4].

6.3.6. Open Subgroup Theorem. Every sufficiently small open subgroup of $S_n$ is cohomologically abelian in the sense that it has the same cohomology as $Z_p^{n^2}$, i.e., an exterior algebra on $n^2$ generators. □

We will give a Hopf algebra theoretic proof of this for a cofinal set of open subgroups, namely the subgroups of elements in $E_n$ congruent to 1 modulo $(S^i)$ for various $i > 0$. The corresponding quotient group (which is finite) is dual the subalgebra of $S(n)$ generated by $\{t_k: k < i\}$. Hence the $i$th subgroup is dual to $S(n)/(t_k: k < i)$, which we denote by $S(n, i)$.

The filtration of 6.3.1 induces one on $S(n, i)$ and analogs of the succeeding four theorems hold for it.

6.3.7. Theorem. If $i \geq n$ and $p > 2$, or $i > n$ and $p = 2$, then

$$H^*(S(n, i)) = E(h_{k,j}: i \leq k < i + n, j \in Z/(n)).$$

Proof. The condition on $i$ is equivalent to $i > n - 1$ and $i > m/2$, where as before $m = pm/(p - 1)$. In the analog of 6.3.3 we have $i, k > m/2$ so $i + k > m$ so the Lie algebra is abelian. We also see that the restriction $\xi$ is injective, so the spectral sequence of 6.3.5 has the $E_2$-term claimed to be $H^*(S(n, i))$. This spectral sequence collapses because $h_{k,j}$ corresponds to $t_{k,j}^p \in S(n, i)$, which is primitive for each $k$ and $j$. □

Proof of 6.2.10(a). Let $A(i)$ be the Hopf algebra corresponding to the quotient of $S_n$ by the $i$th congruence subgroup, so we have a Hopf algebra extension (A1.1.15)

$$A(i) \rightarrow S(n) \rightarrow S(n, i).$$

The corresponding Cartan–Eilenberg spectral sequence (A1.3.14) has

$$E_2 = \text{Ext}_{A(i)}(F_p, H^*(S(n, i)))$$

and converges to $H^*(S(n))$ with $d_r: E_r \rightarrow E_{r+r+1}$. Each $E_r$-term is finitely generated since $A(i)$ and $H^*(S(n, i))$ are finite-dimensional for $i > m/2$. Moreover, $E_{n^2} = E_\infty$, so $E_\infty$ and $H^*(S(n))$ are finitely generated. □
Now we continue with the computation of \( H^*(S(n)) \). Theorem 6.3.5 indicates
the necessity of computing \( H^*(L(n,k)) \) for \( k \leq m \), and this may be done with the
Koszul complex, i.e.,

6.3.8. **Theorem.** \( H^*(L(n,k)) \) for \( k \leq m \) is the cohomology of the exterior
complex \( E(h_{i,j}) \) on one-dimensional generators \( h_{i,j} \) with \( i \leq k \) and \( j \in \mathbb{Z}/(n) \), with
coboundary
\[
d(h_{i,j}) = \sum_{0 < s < i} h_{s,j} h_{i-s,s+j}.
\]
The element \( h_{i,j} \) corresponds to the element \( x_{i,j} \) and therefore has filtration degree
\( i \) and internal degree \( 2p_j(p^i - 1) \).

**Proof.** This follows from standard facts about the cohomology of Lie algebras
(Cartan and Eilenberg [1, XII, Section 7]). \( \square \)

Since \( L(n,k) \) is nilpotent its cohomology can be computed with a sequence of
change-of-rings spectral sequences analogous to A1.3.14.

6.3.9. **Theorem.** There are spectral sequences with
\[
E_2 = E(h_{k,j}) \otimes H^*(L(n,k-1)) \Rightarrow H^*(L(n,k))
\]
and \( E_3 = E_\infty \).

**Proof.** The spectral sequence is that of Hochschild–Serre (see Cartan and
Eilenberg [1, pp. 349–351] for the extension of Lie algebras
\( A(n,k) \to L(n,k) \to L(n,k-1) \)
where \( A(n,k) \) is the abelian Lie algebra on \( x_{k,j} \). Hence \( H^*(A(n,k)) = E(h_{k,j}) \).
The \( E_2 \)-term, \( H^*(L(n,k-1), H^*(A(n,k))) \) is isomorphic to the indicated tensor
product since the extension is central.

For the second statement, recall that the spectral sequence can be constructed
by filtering the complex of 6.3.8 in the obvious way. Inspection of this filtered
complex shows that \( E_3 = E_\infty \). \( \square \)

In addition to the spectral sequence of 6.3.4(a), there is an alternative method of
computing \( H^*\hat{E}_0 S(n)^* \). Define \( \hat{L}(n,k) \) for \( k \leq m \) to be the quotient of \( PE_0 S(n)^* \)
by the restricted sub-Lie algebra generated by the elements \( x_{i,j} \) for \( k < i \leq m \), and
define \( F(n,k) \) to be the kernel of the extension
\[
0 \to F(n,k) \to \hat{L}(n,k) \to \hat{L}(n,k-1) \to 0.
\]
Let \( H^*(\hat{L}(n,k)) \) denote the cohomology of the restricted enveloping algebra of
\( \hat{L}(n,k) \). Then we have

6.3.10. **Theorem.** There are change-of-rings spectral sequences converging to
\( H^*(\hat{L}(n,k)) \) with
\[
E_2 = H^*(F(n,k)) \otimes H^*(\hat{L}(n,k-1))
\]
where
\[
H^*(F(n,k)) = \begin{cases} 
E(h_{k,j}) & \text{for } k > m-n \\
E(h_{k,j}) \otimes P(b_{k,j}) & \text{for } k \leq m-n
\end{cases}
\]
and \( H^*(\hat{L}(n,m)) = H^*(E_0 S(n)^*) \).
PROOF. Again the spectral sequence is that given in Theorem XVI.6.1 of Cartan and Eilenberg [1]. As before, the extension is cocentral, so the \( E_2 \)-term is the indicated tensor product. The structure of \( H^*(F(n,k)) \) follows from 6.3.3 and the last statement is a consequence of 6.3.5.

We begin the computation of \( H^1(S(n)) \) with:

6.3.11. Lemma. \( H^1(E_0 S(n)^*) = H^1(E_0 S(n)) \) is generated by

\[
\zeta_n = \sum_j h_{n,j} \quad \text{and} \quad \rho_n = \sum_j h_{2n,j} \quad \text{for} \ p = 2;
\]

and for \( n > 1 \), \( h_{1,j} \) for each \( j \in \mathbb{Z}/(n) \).

PROOF. By 6.3.4(a) and 6.3.5 \( H^1(E_0 S(n)) = H^1L(n,m) \). The indicated elements are nontrivial cycles by 6.3.8. It follows from 6.3.3 that \( L(n,m) \) can have no other generators since \( [x_{1,j}, x_{i-1,j+1}] = x_{i,j} - \delta_{i+j} x_{i,j+1} \).

In order to pass to \( H^1(S(n)) \) we need to produce primitive elements in \( S(n) \) corresponding to \( \zeta_n \) and \( \rho_n \) (the primitive \( t_1^{p^j} \) corresponds to \( h_{1,j} \)). We will do this with the help of the determinant of a certain matrix. Recall from (6.2.3) that \( S(n) \otimes \mathbb{F}_{p^n} \) was isomorphic to the dual group ring of \( S_n \) which has a certain faithful representation over \( W(\mathbb{F}_{p^n}) \) (6.2.6). The determinant of this representation gave a homomorphism of \( S(n) \) into \( \mathbb{Z}^*_p \), the multiplicative group of units in the \( p \)-adic integers. We will see that in \( H^1 \) this map gives us \( \zeta_n \) and \( \rho_n \).

More precisely, let \( M = (m_{i,j}) \) be the \( n \times n \) matrix over \( \mathbb{Z}_p[t_1, t_2, \ldots]/(t_i - t_i^{p^n}) \) given by

\[
m_{i,j} = \begin{cases} 
\sum_{k \geq 0} p^k p_{n+j-i}^{p^j} & \text{for} \ i \leq j \\
\sum_{k \geq 0} p^{k+1} p_{n+j-i}^{p^j} & \text{for} \ i > j
\end{cases}
\]

where \( t_0 = 1 \).

Now define \( T_n \in S(n) \) to be the \( (p) \) reduction \( p^{-1}(\det M - 1) \) and for \( p = 2 \) define \( U_n \in S(n) \) to be the \( (2) \) reduction of \( \frac{1}{8}(\det M^2 - 1) \). Then we have

6.3.12. Theorem. The elements \( T_n \in S(n) \) and, for \( p = 2 \), \( U_n \in S(n) \) are primitive and represent the elements \( \zeta_n \) and \( \rho_n + \zeta_n \in H^1(S(n)) \), respectively. Hence \( H^1(S(n)) \) is generated by these elements and for \( n > 1 \) by the \( h_{1,j} \) for \( j \in \mathbb{Z}/(n) \).

PROOF. The statement that \( T_n \) and \( U_n \) are primitive follows from 6.2.6. That they represent \( \zeta_n \) and \( \rho_n + \zeta_n \) follows from the fact that

\[
T_n \equiv \sum_j t_1^{n^j} \mod \langle t_1, t_2, \ldots, t_{n-1} \rangle
\]

and

\[
U_n \equiv \sum_j t_2^{n^j} + t_n^{2^j} \mod \langle t_1, t_2, \ldots, t_{n-1} \rangle.
\]

\[\square\]

Examples.

\[
T_1 = t_1, \quad U_1 = t_1 + t_2, \quad T_2 = t_2 + t_1^{p^2} - t_1^{1+p},
\]

\[
U_2 = t_4 + t_2^2 + t_1^2 t_3 + t_1^2 t_2 + t_1^3 t_2 + t_1^2 t_3,
\]
and
\[ T_3 = t_3 + t_3^p + P^2 + t^{1+p^2} - t_1t_2 - t_1^p - t_2^p - t_2. \]

Moreira [1, 3] has found primitive elements in \( BP_*(BP)/I_n \) which reduce to our \( T_n. \) The following result is a corollary of 6.2.7.

6.3.13. **Proposition.** If \( p \nmid n, \) then \( H^*(S(n)) \) decomposes as a tensor product of an appropriate subalgebra with \( E(\zeta_n) \) for \( p > 2 \) and \( P(\zeta_n) \otimes E(\rho) \) for \( p = 2. \) \( \square \)

We now turn to the computation of \( H^2(S(n)) \) for \( n > 2. \) We will compute all of \( H^*(S(n)) \) for \( n = 2 \) below.

6.3.14. **Theorem.** Let \( n > 2 \)
   (a) For \( p = 2, \) \( H^2(S(n)) \) is generated as a vector space by the elements \( \zeta_n^2, \rho_n, \zeta_n, \zeta_nh_{1,j}, \rho_n h_{1,j}, \) and \( h_{1,i}h_{1,j} \) for \( i \neq j \pm 1, \) where \( h_{1,i}h_{1,j} = h_{1,i}h_{1,j} \) and \( h_{1,i}^2 \neq 0. \)
   (b) For \( p > 2, \) \( H^2(S(n)) \) is generated by the elements
\[ \zeta_n h_{1,i}, h_{1,i}, g_i = h_{1,i}h_{1,i+1}h_{1,i}, \text{ and } k_i = h_{1,i+1}h_{1,i+1}h_{1,i} \]
and \( h_{1,i}h_{1,j} \) for \( i \neq j \pm 1, \) where \( h_{1,i}h_{1,j} + h_{1,j}h_{1,i} = 0. \) \( \square \)

Both statements require a sequence of lemmas. We treat the case \( p = 2 \) first.

6.3.15. **Lemma.** Let \( p = 2 \) and \( n > 2. \)
   (a) \( H^1(L(n,2)) \) is generated by \( h_{1,i} \) for \( i \in \mathbb{Z}/(n). \)
   (b) \( H^2(L(n,2)) \) is generated by the elements \( h_{1,i}h_{1,j} \) for \( i \neq j \pm 1, \) \( g_i, k_i, \) and \( e_{3,i} = h_{1,i}h_{1,i+1}h_{1,i+2}. \) The latter elements are represented by \( h_{1,i}h_{2,i}, \)
\( h_{1,i+1}h_{2,i}, \) and \( h_{1,i}h_{2,i+1} + h_{2,i}h_{1,i+2}, \) respectively.
   (c) \( e_{3,i}h_{1,i+1} = h_{1,i}e_{3,i+1} + e_{3,i}h_{1,i+1} + 0, \) and these are the only relations among the elements \( h_{1,i}e_{3,j}. \)

**Proof.** We use the spectral sequence of 6.3.9 with \( E_2 = E(h_{1,i}, h_{2,i}) \) and \( d_2(h_{2,i}) = h_{1,i}h_{1,i+1}. \) All three statements can be verified by inspection. \( \square \)

6.3.16. **Lemma.** Let \( p = 2, n > 2, \) and \( 2 < k \leq 2n. \)
   (a) \( H^1(L(n,k)) \) is generated by the elements \( h_{1,i} \) along with \( \zeta_n \) for \( k \geq n \) and \( \rho_n \) for \( k = 2n. \)
   (b) \( H^2(L(n,k)) \) is generated by products of elements in \( H^1(L(n,k)) \) subject to \( h_{1,i}h_{1,i+1} = 0, \) along with
\[ g_i = h_{1,i}h_{1,i+1}, \quad k_i = h_{1,i}h_{1,i+1}h_{1,i+2}, \quad \alpha_i = h_{1,i}h_{1,i+1}h_{1,i+2}, \]
\[ \text{and } e_{k+1,i} = h_{1,i}h_{1,i+1}, \ldots, h_{1,i+k}. \]
The last two families of elements can be represented by \( h_{3,i}h_{1,i+1} + h_{2,i}h_{2,i+1} \) and \( h_{3,i}h_{1,i+1} + h_{2,i}h_{2,i+1} + h_{3,i}h_{1,i+1}h_{1,i+k} = 0 \) and no other relations hold among products of the \( e_{k+1,i} \) with elements of \( H^1. \)

**Proof.** Again we use 6.3.9 and argue by induction on \( k, \) using 6.3.15 to start the induction. We have \( E_2 = E(h_{k,i}) \otimes H^*(L(n,k-1)) \) with \( d_2(h_{k,i}) = e_{k,i}. \) The existence of the \( \alpha_i, \) follows from the relation \( e_{3,i}h_{1,i+1} = 0 \) in \( H^3(L(n,2)) \) and that of \( e_{k+1,i} \) from \( h_{1,i}e_{k+1,i}h_{1,i+k} = 0 \) in \( H^3(L(n,k-1)). \) The relation \( (c) \) for \( k < 2 \) is formal; it follows from a Massey product identity A1.4.6 or can be verified by
direct calculation in the complex of 6.3.8. No combination of these products can be in the image of $d_2$ for degree reasons.

6.3.17. Let $p = 2$ and $n > 2$. Then $H^2(E_0S(n)^*)$ is generated by the elements $\rho_n\zeta_n$, $\rho_n h_{i,i}$, $\zeta_n h_{1,i}$, $h_{1,i} h_{1,j}$ for $i \neq j \pm 1$, $\alpha_i$, and $h^2_{i,j} = b_{i,j}$ for $1 \leq i \leq n, j \in \mathbb{Z}/(n)$.

**Proof.** We use the modified first May spectral sequence of 6.3.5. We have $m = 2n$ and $H^2(L(n,m))$ is given by 6.3.16. By easy direct computation one sees that $d_2(g_i) = h_{1,i} b_{1,i+1}$ and $d_2(k_i) = h_{1,i} b_{1,i+1}$. We will show that $d_2(e_{2n+1,i}) = h_{1,i} b_{n,i} + h_{1,i+n} b_{n,i-1}$.

\[ \Delta(t_{2n+1}) = \sum t_j \otimes t^{p_j}_{2n+1-j} + b_{n+1,n-1} \]

modulo terms of lower filtration by 4.3.15. Then by 4.3.22

\[ d(b_{n+1,n-1}) = t_1 \otimes b_{n,n} + b_{n,n-1} \otimes t_1 \]

modulo terms of lower filtration and the nontriviality of $d_2(e_{2n+1,i})$ follows.

**Proof of 6.3.14(a).** We now consider the second May spectral sequence (6.3.4(b)). By 4.3.22 we have $d_2(h_{i,j}) = h_{1,j} b_{1,j-1} + h_{1,j} b_{1,j-1} \neq 0$ for $i > 1$. The remaining elements of $H^2E_0S(n)$ survive either for degree reasons or by 6.3.12.

For $p > 2$ we need an analogous sequence of lemmas. We leave the proofs to the reader.

6.3.18. **Lemma.** Let $n > 2$ and $p > 2$.

(a) $H^1(L(n,2))$ is generated by $h_{1,i}$.

(b) $H^2(L(n,2))$ is generated by the elements $h_{1,i} h_{1,j}$ (with $h_{1,i} h_{1,i+1} = 0$).

(c) The only relations among the elements $h_{1,i} e_{3,i}$ are $h_{1,i} e_{3,i+1} - e_{3,i} h_{1,i+3} = 0$.

6.3.19. **Lemma.** Let $n > 2$, $p > 2$, and $2 < k \leq m$. Then

(a) $H^1(L(n,k))$ is generated by $h_{1,i}$ and, for $k \geq n$, $\zeta_n$.

(b) $H^2(L(n,k))$ is generated by $h_{1,i} h_{1,j}$ (with $h_{1,i} h_{1,i+1} = 0$), $g_i$, $h_i$, $e_{k+1,i} = \sum_{0 < j < k+1} h_{j,i} h_{k+1-j,i+j}$.

(c) The only relations among products of elements in $H^1$ with the $e_{k+1,i}$ are $h_{1,i} e_{k+1,i+1} - e_{k+1,i} h_{1,k+1} = 0$.

6.3.20. **Lemma.** Let $n > 2$ and $p > 2$. Then $H^2(E_0S(n)^*)$ is generated by the elements $b_{i,j}$ for $i \leq m - n$ and by the elements of $H^2(L(n,m))$.

**Proof of 6.3.14(b).** Again we look at the spectral sequence of 6.3.4(b). By arguments similar to those for $p = 2$ one can show that

\[ d_p(b_{i,j}) = h_{1,i+j} b_{1,i} - h_{1,j+1} b_{1,j+1} \]

and

\[ d_s(e_{m+1,i}) = h_{1,m+1+i-n} b_{n-n,i-1} - h_{1,j} b_{m-n,j} \]

where $s = 1 + pn - (p - 1)m$, and the remaining elements of $H^2(E_0S(n)^*)$ survive as before.
Now we will compute $H^*(S(n))$ at all primes for $n \leq 2$ and at $p > 3$ for $n = 3$.

6.3.21. Theorem.
(a) $H^*(S(1)) = P(h_{1,0}) \otimes E(p_1)$ for $p = 2$;
(b) $H^*(S(1)) = E(h_{1,0})$ for $p > 2$

[Note that $S(1)$ is commutative and that $\zeta_1 = h_{1,0}$].

Proof. This follows immediately from 6.3.3, 6.3.5, and routine calculation. □

6.3.22. Theorem. For $p > 3$, $H^*(S(2))$ is the tensor product of $E(\zeta_2)$ with the subalgebra with basis $\{1, h_{1,0}, h_{1,1}, g_0, g_1, g_0h_{1,1}\}$ where
\[ g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle, \]
\[ h_{1,0}g_1 = g_0h_{1,1}, \quad h_{1,0}g_0 = h_{1,1}g_1 = 0, \]
and
\[ h_{1,0}h_{1,1} = h_{1,0}^2 = h_{1,1}^2 = 0. \]

In particular, the Poincaré series is $(1 + t)^2(1 + t + t^2)$.

Proof. The computation of $H^*(L(2,2))$ by 6.3.8 or 6.3.9 is elementary, and there are no algebra extension problems for the spectral sequences of 6.3.9 or 6.3.4(b). □

We will now compute $H^*(S(2))$ for $p = 3$. Our description of it in the first edition was incorrect, as was pointed out by Henn [1]. The computation given here is influenced by Henn but self-contained. Henn showed that there are two conjugacy classes of subgroups of order 3 in the group $S_2$. In each case the centralizer is the group of units congruent to one modulo the maximal ideal in the ring of integers of an embedded copy of the field $K = \mathbb{Q}_3[\zeta]$, where $\zeta$ is a primitive cube root of unity.

Let $C_1$ and $C_2$ denote these two centralizers. Henn showed that the resulting map $H^*(S_2) \to H^*(C_1) \oplus H^*(C_2)$ is a monomorphism.

We will describe this map in Hopf algebraic terms. Choose a fourth root of unity $i \in \mathbb{F}_9$, let $a = \pm i$, and consider the two quotients
\[ \overline{S(2)}_+ = S(1, 1)_a, \quad \text{and} \quad \overline{S(2)}_- = S(1, 1)_{-a}, \]
where $S(1, 1)_a$ is the quotient of $S(2) \otimes \mathbb{F}_9$ described in 6.2.13. Henn’s map is presumably equivalent to

(6.3.23) $H^*(S(2)) \otimes \mathbb{F}_9 \to H^*(\overline{S(2)}_+) \oplus H^*(\overline{S(2)}_-)$.

In any case we will show that this map is a monomorphism.

We have the following reduced coproducts in $\overline{S(2)}_\pm$.
\[ \bar{t}_1 \mapsto 0 \]
\[ \bar{t}_2 \mapsto a\bar{t}_1 \otimes \bar{t}_1 \]
\[ \bar{t}_3 \mapsto \bar{t}_1 \otimes \bar{t}_2 + \bar{t}_2 \otimes \bar{t}_1 - a^3(\bar{t}_1^2 \otimes \bar{t}_1 + \bar{t}_1 \otimes \bar{t}_1^2) \]
It follows that $\bar{t}_2 + a_1^2$ and $\bar{t}_3 - \bar{t}_1 \bar{t}_2$ are primitive. The filtration of 6.3.1 induces one on $S(2)_{\pm}$, and the methods of this section lead to

$$H^*(S(2))_{\pm} = E(\bar{t}_1,0, \bar{t}_2,0, \bar{t}_3,0) \otimes P(\bar{b}_1,0)$$

with the evident notation.

6.3.24. Theorem. For $p = 3$, $H^*(S(2))$ is a free module over

$$E(\zeta_2) \otimes P(b_1,0)$$

on the generators

$$\{1, h_{1,0}, h_{1,1}, b_{1,1}, \xi, a_0, a_1, b_{1,1}\xi\},$$

where the elements $\xi \in H^2$ and $a_0, a_1 \in H^3$ will be defined below. The algebra structure is indicated in the following multiplication table.

<table>
<thead>
<tr>
<th></th>
<th>$h_{1,0}$</th>
<th>$h_{1,1}$</th>
<th>$b_{1,1}$</th>
<th>$\xi$</th>
<th>$a_0$</th>
<th>$a_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{1,0}$</td>
<td>0</td>
<td>0</td>
<td>$-b_{1,0}h_{1,1}$</td>
<td>0</td>
<td>$-b_{1,1}\xi$</td>
<td>$-b_{1,0}\xi$</td>
</tr>
<tr>
<td>$h_{1,1}$</td>
<td>0</td>
<td>0</td>
<td>$b_{1,0}h_{1,0}$</td>
<td>0</td>
<td>$-b_{1,0}\xi$</td>
<td>$b_{1,1}\xi$</td>
</tr>
<tr>
<td>$b_{1,1}$</td>
<td>$-b_{1,0}h_{1,1}$</td>
<td>$b_{1,0}h_{1,0}$</td>
<td>0</td>
<td>$-b_{1,0}a_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0</td>
<td>$b_{1,1}\xi$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$a_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$a_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In particular, the Poincaré series is

$$(1 + t)^2(1 + t^2)/(1 - t).$$

Moreover the map of (6.3.23) is a monomorphism.

Proof. Our basic tools are the spectral sequences of 6.3.10 and some Massey product identities from A1.4. We have $H^*(\tilde{L}(2,1)) \cong E(h_{1,0}, h_{1,1}) \otimes P(b_{1,0}, b_{1,1})$, and a spectral sequence converging to $H^*(\tilde{L}(2,2))$ with $E_2 = E(\zeta_2, \eta) \otimes H^*(\tilde{L}(2,1))$, where

$$\zeta_2 = h_{2,0} + h_{2,1}, \quad \eta = h_{2,1} - h_{2,0},$$

$$d_2(\zeta_2) = 0, \quad d_2(\eta) = h_{1,0}h_{1,1},$$

and $E_3 = E_{\infty}$. Hence $E_{\infty}$ is a free module over $E(\zeta_2) \otimes P(b_{1,0}, b_{1,1})$ on generators

$$\{1, h_{1,0}, h_{1,1}, g_0, g_1, h_{1,0}g_1 = h_{1,1}g_0\},$$

where $g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i} \rangle$. This determines the additive structure of $H^*(\tilde{L}(2,2))$, but there are some nontrivial extensions in the multiplicative structure. We know by 6.3.13 that we can factor out $E(\zeta_2)$, and we can write $b_{1,i}$ as the Massey product $\langle h_{1,i}, h_{1,i}, h_{1,i} \rangle$. Then by A1.4.6 we have $h_{1,i}g_i = -b_{1,i}h_{1,i+1}$, $g_i^2 = -b_{1,i}g_{i+1}$, $g_ig_{i+1} = b_{1,i}b_{1,i+1}$. These facts along with the usual $h_{1,i} = h_{1,0}h_{1,1} = 0$ determine $H^*(\tilde{L}(2,2))$ as an algebra.

This algebra structure allows us to embed $H^*(\tilde{L}(2,2))$ in the ring

$$R = E(\zeta_2, h_{1,0}, h_{1,1}) \otimes P(s_0, s_1)/(h_{1,0}h_{1,1}, h_{1,0}s_1 - h_{1,1}s_0)$$

by sending $\zeta_2$ and $h_1, i$ to themselves and

$$b_{1,i} \mapsto -s_i^3,$$
$$g_0 \mapsto s_0^2s_1,$$  
$$g_1 \mapsto s_0s_1^2.$$
Here the cohomological degree of \( s_i \) is 2/3, and \( H^*(\tilde{L}(2, 2)) \) maps isomorphically to the subring of \( R \) consisting of elements of integral cohomological degree.

Next we have the spectral sequence of 6.3.10 converging to

\[
H^*(\tilde{L}(2, 3)) \cong H^*(E^3S(2))
\]

with \( E_2 = E(h_{3,0}, h_{3,1}) \otimes H^*(\tilde{L}(2, 2)) \), and \( d_2(h_{3,i}) = g_i - b_{1,i+1} \). We will see shortly that \( E_3 = E_\infty \) for formal reasons. Tensoring this over \( H^*(\tilde{L}(2, 2)) \) with \( R \) gives a spectral sequence with

\[
E_2 = E(h_{3,0}, h_{3,1}) \otimes R
\]

and

\[
d_2(h_{3,0}) = s_1(s_0^2 + s_1^2)
\]

\[
d_2(h_{3,1}) = s_0(s_0^2 + s_1^2).
\]

This can be simplified by tensoring with \( F_9 \) (which contains \( i = \sqrt{-1} \)) and defining

\[
x_0 = h_{1,0} + ih_{1,1} \quad x_1 = h_{1,0} - ih_{1,1}
\]

\[
y_0 = s_0 + is_1 \quad y_1 = s_0 - is_1
\]

\[
z_0 = ih_{3,0} + h_{3,1} \quad z_1 = -ih_{3,0} + h_{3,1}
\]

The Galois group of \( F_9 \) over \( F_3 \) acts here by conjugating scalars and permuting the two subscripts. Then we have

\[
R \otimes F_9 = E(\zeta_2, x_0, x_1) \otimes P(y_0, y_1)/(x_0 x_1 - x_1 y_0),
\]

where the cohomological degrees of \( x_i \) and \( y_i \) are 1 and 2/3 respectively. In the spectral sequence we have

\[
d_2(z_0) = y_0^2 y_1 \quad \text{and} \quad d_2(z_1) = y_0 y_1^2.
\]

The image of \( H^*(\tilde{L}(2, 2)) \otimes F_9 \) in \( R \otimes F_9 \) is a free module over the ring

\[
B = E(\zeta_2) \otimes P(y_0^2, y_1^3)
\]

on the following set of six generators.

\[
C = \{ 1, x_0, x_1, y_0^2 y_1, y_0 y_1^2, x_0 y_0 y_1^2 = x_1 y_0^3 y_1 \}
\]

Hence the image of \( E(h_{3,0}, h_{3,1}) \otimes H^*(\tilde{L}(2, 2)) \otimes F_9 \) is a free \( B \)-module on the set

\[
\{ 1, z_0, z_1, z_0 z_1 \} \otimes C,
\]

but it is convenient to replace this basis by the set of elements listed in the following table.

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( z_0 )</th>
<th>( z_1 )</th>
<th>( z_0 z_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>( x_0 z_0 )</td>
<td>( \beta = x_0 z_1 - x_1 z_0 = -x_0 z_0 z_1 )</td>
<td></td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( \delta = x_1 z_0 - x_0 z_1 )</td>
<td>( x_1 z_1 )</td>
<td>( x_1 z_0 z_1 )</td>
</tr>
<tr>
<td>( y_0 y_1 )</td>
<td>( \alpha_1 = y_0^2 y_1 z_0 - y_0 y_1^2 z_0 )</td>
<td>( \xi = y_0^2 y_1 z_1 - y_0 y_1^2 z_0 )</td>
<td>( y_0^2 y_1 z_0 z_1 )</td>
</tr>
<tr>
<td>( y_0 y_1^2 )</td>
<td>( \gamma = -y_0 y_1^2 z_0 - y_0 y_1 z_1 )</td>
<td>( \alpha_0 = y_0 y_1^2 z_1 - y_0 y_1 z_0 )</td>
<td>( -y_0 y_1^2 z_0 z_1 )</td>
</tr>
<tr>
<td>( x_0 y_0 y_1^2 )</td>
<td>( -x_0 \xi )</td>
<td>( x_1 \xi )</td>
<td>( x_0 y_0 y_1^2 z_0 z_1 )</td>
</tr>
</tbody>
</table>

This basis is Galois invariant up to sign, i.e., the Galois image of each basis element is another basis element. The elements \( 1, x_0 y_0 y_1^2, \delta, \) and \( \gamma \) are self-conjugate, while \( \beta, \xi, z_0 z_1 \) and \( x_0 y_0 y_1^2 z_0 z_1 \) are antiself-conjugate. The remaining elements form eight conjugate pairs.
In the spectral sequence the following twelve differentials (listed as six Poincaré dual pairs) are easily derived from (6.3.25) and account for each of these 24 basis elements.

\[
\begin{align*}
    d_2(z_0) &= y_0^3y_1^3, & d_2(x_0z_0) &= x_1\varepsilon \\
    d_2(z_1) &= y_0y_1^2, & d_2(-x_0z_0z_1) &= -x_0\varepsilon \\
    d_2(z_0z_1) &= \varepsilon, & d_2(\delta) &= x_0y_0^3y_1^2 \\
    d_2(x_0z_0) &= y_0^2(x_1), & d_2(y_0^2y_1^2z_0z_1) &= y_0^2(\alpha_0) \\
    d_2(x_1z_1) &= y_1^2(x_0), & d_2(-y_0^2y_1^2z_0z_1) &= y_1^2(\alpha_1) \\
    d_2(\gamma) &= y_0^3y_1^3(1), & d_2(x_0y_0^2y_1^2z_0z_1) &= y_0^3y_1^3(\beta)
\end{align*}
\]

The spectral sequence collapses from $E_3$ since there are no elements in $E_3^{*,t}$ for $t > 1$. The image of $H^*(L(2,3)) \otimes F_9$ in the $E_\infty$-term is the $B$-module generated by

\[
\{1, x_0, x_1, \alpha_0, \alpha_1, \beta\}
\]

subject to the module relations

\[
\begin{align*}
    y_0^3y_1^3(1) &= 0, & y_0^3y_1^3(\beta) &= 0, \\
    y_0^3(x_1) &= 0, & y_0^3(\alpha_0) &= 0, \\
    y_1^3(x_0) &= 0, & y_1^3(\alpha_1) &= 0.
\end{align*}
\]

The only nontrivial products among these six elements are

\[
x_0\alpha_1 = -y_0^3\beta \quad \text{and} \quad x_1\alpha_0 = y_1^3\beta.
\]

Equivalently the image is the free module over $E(\zeta_2) \otimes P(y_0^3 + y_1^3)$ on the eight generators

\[
\{1, x_0, x_1, y_0^3, \beta, \alpha_0, \alpha_1, y_1^3\beta\}
\]

with suitable algebra relations.

It follows that $H^*(E^0S(2))$ itself is a free module over $E(\zeta_2) \otimes P(b_{1,0})$ on the eight generators

\[
\{1, h_{1,0}, h_{1,1}, b_{1,1}, \xi, a_0, a_1, b_{1,1}\xi\},
\]

where

\[
\xi = i\beta, \quad a_0 = \alpha_0 + \alpha_1, \quad \text{and} \quad a_1 = i(\alpha_0 - \alpha_1).
\]

It also follows that $E^0H^*(S(2))$ has the relations stated in the theorem. The absence of nontrivial multiplicative extensions in $H^*(S(2))$ will follow from the fact that the map of (6.3.23) is monomorphic and there are no extensions in its target.

Now we will determine the images of the elements of (6.3.26) under the map of (6.3.23). Recall that

\[
H^*(S(2)) \otimes F_9 \to E(\overline{h}_{3,0}) \otimes \overline{R}_+ \oplus E(\overline{h}_{3,0}) \otimes \overline{R}_-
\]

As before it is convenient to adjoin a cube root $\xi_0$ of $-\overline{h}_{1,0}$ and let

\[
\overline{R}_\pm = E(\overline{h}_{1,0}, \overline{h}_{2,0}) \otimes P(\xi_0).
\]

The map

\[
H^*(S(2)) \otimes F_9 \to E(\overline{h}_{3,0}) \otimes \overline{R}_+ \oplus E(\overline{h}_{3,0}) \otimes \overline{R}_-
\]
behaves as follows.
\[
\begin{align*}
 x_0 & \mapsto (0, -\bar{h}_{1,0}) & x_1 & \mapsto (-\bar{h}_{1,0}, 0) \\
y_0 & \mapsto (0, -\bar{s}_0) & y_1 & \mapsto (-\bar{s}_0, 0) \\
z_0 & \mapsto (-i\bar{h}_{3,0}, 0) & z_1 & \mapsto (0, i\bar{h}_{3,0}) \\
\beta & \mapsto (-i\bar{h}_{1,0}\bar{h}_{3,0}, -i\bar{h}_{1,0}\bar{h}_{3,0}) \\
\alpha_0 & \mapsto (i\bar{s}_0\bar{h}_{3,0}, 0) & \alpha_1 & \mapsto (0, -i\bar{s}_0\bar{h}_{3,0})
\end{align*}
\]

It follows that Henn’s map is a monomorphism. \hfill \square

We now turn to the case \( n = p = 2 \). We will only compute \( E^0 H^*(S(2)) \), so there will be some ambiguity in the multiplicative structure of \( H^*(S(2)) \). In order to state our result we need to define some classes. Recall (6.3.12) that \( H^1(S(2)) \) is the \( \mathbb{F}_2 \)-vector space generated by \( h_{1,0}, h_{1,1}, \zeta_2 \) and \( \rho_2 \). Let
\[
\alpha_0 \in \langle \zeta_2, h_{1,0}, h_{1,1} \rangle, \quad \beta \in \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1} \rangle, \quad g = \langle h, h^2, h, h^2 \rangle,
\]
where \( h = h_{1,0} + h_{1,1}, \ \bar{x} = \langle x, h, h^2 \rangle \) for \( x = \zeta_2, \alpha_0, \zeta_2^2 \), and \( \alpha_0\zeta_2 \) (more precise definitions of \( \alpha_0 \) and \( \beta \) will be given in the proof).

6.3.27. THEOREM. \( E^0 H^*(S(2)) \) for \( p = 2 \) is a free module over \( P(g) \otimes E(\rho_2) \) on 20 generators: 1, \( h_{1,0}, h_{1,1}, h_{1,0}^2, h_{1,1}^2, h_{1,0}^3, h_{1,1}^3, \beta, \beta h_{1,0}, \beta h_{1,1}, \beta h_{1,0}^2, \beta h_{1,1}^2, \beta h_{1,0}^3, \beta h_{1,1}^3, \zeta_2, \alpha_0, \zeta_2^2, \alpha_0\zeta_2, \tilde{\zeta}_2, \alpha_0\tilde{\zeta}_2, \alpha_0\tilde{\zeta}_2^2, \alpha_0\tilde{\zeta}_2^2, \alpha_0\tilde{\zeta}_2^2 \), where \( \alpha_0 \in H^2(S(2)) \) and has filtration degree 4, \( \beta \in H^3(S(2)) \) and has filtration degree 8, \( g \in H^4(S(2)) \) and has filtration degree 8, and the cohomological and filtration degrees of \( \bar{x} \) exceed those of \( x \) by 2 and 4, respectively. Moreover \( h_{1,0}^3 = h_{1,1}^3, \alpha_0^2 = \zeta_2^2, \) and all other products are zero. The Poincaré series is \( (1 + t)^2(1 - t^5)/(1 - t)^2(1 + t^2) \).

PROOF. We will use the same notation for corresponding classes in the various cohomology groups we will be considering along the way.

Again our basic tool is 6.3.10. It follows from 6.3.5 that \( H^*(E_0 S(2)^*) \) is the cohomology of the complex
\[
P(h_{1,0}, h_{1,1}, \zeta_2, h_{2,0}) \otimes E(h_{3,0}, h_{3,1}, \rho_2, h_{4,0})
\]
with
\[
d(h_{1,1}) = d(\zeta_2) = d(\rho_2) = 0, \\
d(h_{3,0}) = h_{1,0}\zeta_2, \quad d(h_{2,0}) = h_{1,0}h_{1,1},
\]
and
\[
d(h_{4,0}) = h_{1,0}h_{3,1} + h_{1,1}h_{3,0} + \zeta_2^2.
\]

This fact will enable us to solve the algebra extension problems in the spectral sequences of 6.3.10.

For \( H^*(\bar{L}(2, 2)) \) we have a spectral sequence with \( E_2 = P(h_{1,0}, h_{1,1}, \zeta_2, h_{2,0}) \) with \( d_2(\zeta_2) = 0 \) and \( d_2(h_{2,0}) = h_{1,0}h_{1,1} \). It follows easily that
\[
H^*(\bar{L}(2, 2)) = P(h_{1,0}, h_{1,1}, \zeta_2, b_{2,0})/(h_{1,0}h_{1,1})
\]
where \( b_{2,0} = h_{2,0}^2 = \langle h_{1,0}, h_{1,1}, h_{1,0}, h_{1,1} \rangle \).

For \( H^*(\bar{L}(2, 3)) \) we have a spectral sequence with
\[
E_2 = E(h_{3,0}, h_{3,1}) \otimes H^*(\bar{L}(2, 2))
\]
and \( d_2(h_{3,i}) = h_{1,i} \zeta_2 \). Let
\[
\alpha_i = h_{1,i+1} h_{3,i} + \zeta_2 h_{2,i} \in \langle \zeta_2, h_{1,i}, h_{1,i+1} \rangle.
\]
Then \( H^*(\tilde{L}(2,3)) \) as a module over \( H^*(\tilde{L}(2,2)) \) is generated by \( 1, \alpha_0, \) and \( \alpha_1 \) with
\[
\zeta_2 h_{1,i} = \zeta_2 (\alpha_0 + \alpha_1 + \zeta_2^2) = h_{1,i} \alpha_i = \zeta_2 h_{1,i+1} \alpha_i = 0 \]
and
\[
\alpha_0^2 = \zeta_2^2 h_{2,0}, \quad \alpha_1^2 = \zeta_2^3 (\zeta_2^2 + h_{2,0}), \quad \alpha_0 \alpha_1 = \zeta_2^3 (\alpha_0 + h_{2,0}).
\]
The Poincaré series for \( H^*(\tilde{L}(2,3)) \) is \((1 + t + t^2)/(1 - t^2)\).

For \( H^*(\tilde{L}(2,4)) \) we have a spectral sequence with
\[
E_2 = E(h_{4,0}, \rho_2) \otimes H^*(\tilde{L}(2,3)),
\]
d\((\rho_2) = 0, \) and \( d_2(h_{4,0}) = \alpha_0 + \alpha_1. \) Define \( \beta \in H^3(\tilde{L}(2,4)) \) by
\[
\beta = h_{4,0} (\alpha_0 + \alpha_1 + \zeta_2^3) + \zeta_2 h_{3,0} h_{3,1} \in \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1} \rangle.
\]
Then \( H^*(\tilde{L}(2,4)) \) is a free module over \( E(\rho_2) \otimes P(b_{2,0}) \) on generators \( 1, h_{1,1}^1, \zeta_2, \zeta_2^2, \alpha_0, \alpha_0 \zeta_2, \beta, \) and \( \beta h_{1,1}^1, \) where \( t > 0. \) As a module over \( H^*(\tilde{L}(2,3)) \otimes E(\rho_2) \) it is generated by \( 1 \) and \( \beta, \) with \( (\alpha_0 + \alpha_1) 1 = \zeta_2^3 (1) = \alpha_0 \zeta_2^3 (1) = 0. \) To solve the algebra extension problem we observe that \( \beta \zeta_2 = 0 \) for degree reasons; \( \beta \alpha_i = \beta (\zeta_2, h_{1,i}, h_{1,i+1}) \) is \( \langle \beta, \zeta_2, h_{1,i} \rangle h_{1,i+1} = 0 \) since \( \langle \beta, \zeta_2, h_{1,i} \rangle = 0 \) for degree reasons; and \( E(\rho_2) \) splits off multiplicatively by the remarks at the beginning of the proof.

This completes the computation of \( H^*(E_0 S(2)^*) \). Its Poincaré series is \((1 + t)^2/(1 - t)^2. \) We now use the second May spectral sequence [6.3.4(b)] to pass to \( E^3 H^*(S(2)) \). \( H^*(E_0 S(2)^*) \) is generated as an algebra by the elements \( h_{1,0}, h_{1,1}, \zeta_2, \rho_2, \alpha_0, b_{2,0}, \) and \( \beta. \) The first four of these are permanent cycles by 6.3.12.

By direct computation in the cobar resolution we have
\[
d(t_3 + t_1 t_2^2) = \zeta_2 \otimes t_1,
\]
so the Massey product for \( \alpha_0 \) is defined in \( H^*(S(2)) \) and the \( \alpha_0 \) is a permanent cycle. We also have
\[
d(t_2 \otimes t_2 + t_1 \otimes t_2 t_2 + t_1 t_2 \otimes t_2^2) = t_1 \otimes t_1 + t_2 \otimes t_2^2 \otimes t_2^2,
\]
so \( d_2(b_{2,0}) = h_{1,0}^2 + h_{1,1}^2. \) Inspection of the \( E_3 \) term shows that \( b_{2,0} = \langle h, h^2, h, h^2 \rangle, \) (where \( h = h_{1,0} + h_{1,1} \)) is a permanent cycle for degree reasons.

We now show that \( \beta = \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1} \rangle \) is a permanent cycle by showing that its Massey product expression is defined in \( E^3 H^*(S(2)) \). The products \( h_{1,0} \zeta_2 \) and \( \zeta_2^3 h_{1,1} \) are zero by 6.3.28 and we have
\[
d(t_3 + t_1 t_2^2 + T_2 \otimes t_4 + T_2 \otimes t_2^2 + T_2 \otimes t_1^2 (1 + t_2 + t_2^2)) = T_2 \otimes T_2 \otimes T_2,
\]
where \( t_3 = t_3 + t_1 t_2^2 \) and \( T_2 = t_2 + t_2^2 + t_1^2, \) so \( \zeta_2^3 = 0 \) in \( H^*(S(2)) \). Inspection of \( H^3(E_0 S(2)^*) \) shows there are no elements of internal degree 2 or 4 and filtration degree \( > 7, \) so the triple products \( \langle h_{1,0}, \zeta_2, \zeta_2^3 \rangle \) and \( \langle \zeta_2, \zeta_2^3, h_{1,1} \rangle \) must vanish and \( \beta \) is a permanent cycle.

Now the \( E_3 \) term is a free module over \( E(\rho_2) \otimes P(b_{2,0}^2) \) on 20 generators: \( 1, h_{1,0}, h_{1,1}, h_{2,0}^1, h_{2,1}^1, h_{1,1}^3 = h_{1,0}^3, \) \( \beta, \beta h_{1,0}, \beta h_{1,1}, \beta h_{2,0}^1, \beta h_{2,1}^1, \beta^3, \beta h_{3,0}, \zeta_2, \alpha_0, \zeta_2^2, \alpha_0 \zeta_2, \zeta_2 b_{1,0}, h_{0} b_{2,0}, \zeta_2^3 b_{2,0}, \) \( \zeta_2 \alpha_0 b_{2,0}. \) The last four in the list now have Massey product expressions \( \langle \zeta_2, h, h^2 \rangle, \langle \alpha_0, h, h^2 \rangle, \langle \zeta_2^2, h, h^2 \rangle, \) and \( \langle \alpha_0, \zeta_2, h, h^2 \rangle, \) respectively. These
elements have to be permanent cycles for degree reasons, so \( E_3 = E_\infty \), and we have determined \( E^0 H^*(S(2)) \).

We now describe an alternative method of computing \( H^*(S(2) \otimes \mathbb{F}_4) \), which is quicker than the previous one, but yields less information about the multiplicative structure. By 6.3.4, this group is isomorphic to \( H^*(S_2; \mathbb{F}_4) \), the continuous cohomology of certain 2-adic Lie group with trivial coefficients in \( \mathbb{F}_4 \). \( S_2 \) is the group of units in the degree 4 extension \( E_2 \) of \( \mathbb{Z}_2 \) obtained by adjoining \( \omega \) and \( S \) with \( \omega^2 + \omega + 1 = 0 \) and \( S\omega = \omega^2 S \).

Let \( Q \) denote the quaternion group, i.e., the multiplicative group (with 8 elements) of quaternionic integers of modulus 1.

6.3.30. Proposition. There is a split short exact sequence of groups

\[
(6.3.31) \quad 1 \to G \xrightarrow{i} S_2 \xrightarrow{j} Q \to 1.
\]

The corresponding extension of dual group algebras over is

\[ Q_\ast \xrightarrow{i} S(2) \xrightarrow{j} G_\ast \]

where \( Q_\ast \cong \mathbb{F}_4[x,y]/(x^4 - x, y^2 - y) \) and \( G_\ast \cong S(2)/(t_1, t_2 + \omega t_2^2) \) as algebras where \( j_*(x) = 1 \), \( j_*(y) = \bar{\omega} t_2 + \bar{\omega} x^2 t_2^2 \), and \( \bar{\omega} \) is the residue class of \( \omega \).

Proof. The splitting follows the theory of division algebras over local fields (Cassels and Fröhlich [1, pp. 137–138]) which implies that \( E_2 \otimes \mathbb{Q} \) is isomorphic to the 2-adic quaternions. We leave the remaining details to the reader.

6.3.32. Proposition. In the Cartan–Eilenberg spectral sequence for 6.3.31, \( E_3 = E_\infty \) and we get the same additive structure for \( H^*(S(2)) \) as in 6.3.27.

Proof. We can take \( H^*(G) \otimes H^*(Q) \) as our \( E_1 \)-term. Each term is a free module over \( E(\rho_2) \otimes P(g) \). We leave the evaluation of the differentials to the reader.

Finally, we consider the case \( n = 3 \) and \( p \geq 5 \). We will not make any attempt to describe the multiplicative structure. An explicit basis of \( E^0 H^*(S(3)) \) will be given in the proof, from which the multiplicity can be read off by the interested reader. It seems unlikely that there are any nontrivial multiplicative extensions.

6.3.34. Theorem. For \( p \geq 5 \), \( H^*(S(3)) \) has the following Poincaré series:

\[
(1 + t)^3(1 + t + 6t^2 + 3t^3 + 6t^4 + t^5 + t^6).
\]

Proof. We use the spectral sequences of 6.3.9 to compute \( H^*(L(3,2)) \) and \( H^*(L(3,3)) \). For the former the \( E_2 \)-term is \( H(h_{1,i}) \otimes E(h_{2,i}) \) with \( i \in \mathbb{Z}/(3) \), \( d_2(h_{1,i}) = 0 \) and \( d_2(h_{2,i}) = h_{1,i} h_{1,i+1} \). The Poincaré series for \( H^*(L(3,2)) \) is \((1 + t)^2(1 + t + 5t^2 + t^3 + t^4)\) and it is generated as a vector space by the following elements and their Poincaré duals: \( 1, h_{1,i}, g_i = h_{1,i} h_{2,i}, k_i = h_{2,i} h_{1,i+1}, e_{3,i} = \cdots \)
$h_1, h_{2i+1} + h_2, h_{1, i+2}$ (where $\sum_i e_{3,i} = 0$), $g_i, h_{1, i+1} = h_1, k_i = h_1, h_2, h_{1, i+1}$, and $h_1, e_{3,i} = g_i, h_{1, i+2} = h_1, h_2, h_{1, i+2}$.

For $H^*(L(3, 3))$ we have $E_2 = E(h_3,i \otimes H^*(L(3, 2)))$ with $d_2(h_3,i) = e_{3,i}$, so $d_2(\sum h_{3,i}) = 0$. $H^*(L(3, 3))$ has the indicated Poincaré series and is a free module over $E(\zeta_3)$, where $\zeta_3 = \sum h_{3,i}$, on the following 38 elements and the duals of their products with $\zeta_3$:

$$1, \ h_1, i, \ g_i, \ k_i, \ b_i, h_{1, i+2} = h_1, h_3, i + h_2, h_{i+2} + h_3, h_{1,i},$$
$$g_i, h_{1, i+1} = h_1, i, \ k_i, \ h_1, h_2, h_{2, i+2}, \ h_1, h_2, h_{2, i+1} + h_1, h_{1, i+1} h_3, i,$$
$$h_{1, i}, h_2, h_3, i, \ h_{1, i}, h_{2, i+2} h_3, i + 1, \ \sum \theta i, h_{2, i+1} - h_{1, i+1} h_{2, i+2})h_3, i, \ h_1, k_i h_3, j$$

(where $h_1, k_i \sum_j h_3, j$ is divisible by $\zeta_3$), and $h_{1, i+2} h_1, h_{2, i}(h_{3, i} + h_{3, i+1}) \leq h_1, h_2, h_{2, i+2}$.

4. The Odd Primary Kervaire Invariant Elements

The object of this section is to apply the machinery above to show that the Adams–Novikov element $\beta_{p^i/p^i} \in \Ext^2$ (see 5.1.19) is not a permanent cycle for $p > 2$ and $i > 0$. This holds for the corresponding Adams element $b_i$ (4.3.2) for $p > 3$ and $i > 0$; by 5.4.6 we know $\beta_{p^i/p^i}$ maps to $b_i$. The latter corresponds to the secondary cohomology operation associated with the Adem relation $P^{p^i-1}p^i p^i = \cdots$. The analogous relation for $p = 2$ is $Sq^2 Sq^2 = \cdots$, which leads to the element $h_2^2$, which is related to the Kervaire invariant by Browder’s theorem, hence the title of the section. To stress this analogy we will denote $\beta_{p^i/p^i}$ by $\theta_i$.

We know by direct calculation (e.g., 4.4.20) that $\theta_0$ is a permanent cycle corresponding to the first element in coker $J$. By Toda’s theorem (4.4.22) we know $\theta_1$ is not a permanent cycle; instead we have $d_{2p-1}(\theta_1) = \alpha_1 \theta_0^p$ (up to nonzero scalar multiplication) and this is the first nontrivial differential in the Adams–Novikov spectral sequence. Our main result is

6.4.1. Odd Primary Kervaire Invariant Theorem. In the Adams–Novikov spectral sequence for $p > 2$ $d_{2p-1}(\theta_1) = \alpha_1 \theta_0^p$ mod $\ker \theta_0^p$ (up to nonzero scalar multiplication) where $a_i = p(p^i-1)/(p-1)$ and $\alpha_1 \theta_0^p$ is nonzero modulo this indeterminacy.

Our corresponding result about the Adams spectral sequence fails for $p = 3$, where $b_2$ is a permanent cycle even though $b_1$ is not.

6.4.2. Theorem. In the Adams spectral sequence for $p \geq 5$ $b_i$ is not a permanent cycle for $i \geq 1$.

From 6.4.1 we can derive the nonexistence of certain finite complexes which would be useful for constructing homotopy elements with Novikov filtration 2.

6.4.3. Theorem. There is no connective spectrum $X$ such that

$$BP_*(X) = BP_*/(p, v_1^p, v_2^p)$$

for $i > 0$ and $p > 2$.

Proof. Using methods developed by Smith [1], one can show that such an $X$ must be an 8-cell complex and that there must be cofibrations

(i) $\Sigma^{2p(p^i-1)} Y \xrightarrow{f} Y' \to X,$
210 6. MORAVA STABILIZER ALGEBRAS

(ii) \( \Sigma^{2p^{i/p-1}}V(0) \xrightarrow{g} V(0) \to Y \),

(iii) \( \Sigma^{2p^{i/p-1}}V(0) \xrightarrow{g'} V(0) \to Y' \),

where \( V(0) \) is the mod \((p)\) Moore spectrum, \( g \) and \( g' \) induce multiplication by \( v_1^{p^i} \) in \( BP_\ast(V(0)) = BP_\ast/(p) \), and \( f \) induces multiplication by \( v_2^{p^i} \) in

\[ BP_\ast(Y) = BP_\ast/(p, v_1^{p^i}). \]

\( V(0) \) and the maps \( g, g' \) certainly exist; e.g., Smith showed that there is a map \( \alpha : \Sigma^{2(p^i-1)}V(0) \to V(0) \) which includes multiplication by \( v_1 \), hence \( \alpha v_1^{p^j} \) induces multiplication by \( v_1^{p^j} \), but it may not be the only map that does so.

Hence we have to show that the existence of \( f \) leads to a contradiction. Consider the composite

\[ S^{2p^{i/p-1}} \xrightarrow{j} \Sigma^{2p^{i/p-1}}Y \xrightarrow{f} Y' \xrightarrow{k} S^{2+2p^{i/p-1}}, \]

where \( j \) is the inclusion of the bottom cell and \( k \) is the collapse onto the top cell.

We will show that the resulting element in \( \pi_{S^{2p^{i/p-1}-1}}(Y') \) is detected by \( v_1^{p^j} \in \text{Ext}^0(BP_\ast/(p, v_1^{p^j})) \). We know (5.1.19) that

\[ \theta_1 = \delta_0 \delta_1(v_1^{p^j}) \in \text{Ext}^2(BP_\ast) \]

detects the element \( kfj \in \pi_{S^{2p^{i/p-1}-1}}(Y') \).

The statement in 6.4.1 that \( \alpha_1 \theta_1^p \) is nonzero modulo the indeterminacy is a corollary of the following result, which relies heavily on the results of the previous three sections.

6.4.4. Detection Theorem. In the Adams–Novikov \( E_2 \)-term for \( p > 2 \) let \( \theta^I \) be a monomial in the \( \theta_i \). Then each \( \theta^I \) and \( \alpha_1 \theta^I \) is nontrivial.

We are not asserting that these monomials are linearly independent, which indeed they are not. Certain relations among them will be used below to prove 6.4.1. Assuming 6.4.4, we have
Proof of 6.4.1. We begin with a computation in $\text{Ext}(BP_*/(p))$. We use the symbol $\theta_i$ to denote the mod $p$ reduction of the $\theta_i$ defined above in $\text{Ext}(BP_*)$. We also let $h_i$ denote the element $-\lfloor t_i^p \rfloor$. In the cobar construction we have

$$d(t_2) = -[t_1|t_1^p] + v_1 \sum_{0 < j < p} \frac{1}{p} \binom{p}{j} [t_1^j|t_1^{p-j}]$$

so

$$(6.4.5) \quad v_1 \theta_0 = -h_0 h_1.$$ 

May [5] developed a general theory of Steenrod operations which is applicable to this $\text{Ext}$ group (see A1.5). His operations are similar to the classical ones in ordinary cohomology, except for the fact that $P^0 \neq 1$. Rather we have $P^0(h_i) = h_{i+1}$ and $P^0(\theta_i) = \theta_{i+1}$. We also have $\beta P^0(h_i) = \theta_i$, $\beta P^0(\theta_i) = 0$, $P^1(\theta_i) = \theta_i^p$ and the Cartan formula implies that $P^{p^i}(\theta_i) = \theta_i^{p^i+1}$. Applying $\beta P^0$ to (6.4.6) gives

$$(6.4.6) \quad 0 = \theta_0 h_2 - h_1 \theta_1.$$ 

If we apply the operation $P^{p^i-1} P^{p^i-2} \cdots P^1$ to (6.4.5) we get

$$(6.4.7) \quad h_{1+i} \theta_i^p = h_{2+i} \theta_0^p.$$ 

Now associated with the short exact sequence

$$0 \to BP_* \xrightarrow{\varphi} BP_* \to BP_*/(p) \to 0$$ 

there is a connecting homomorphism

$$\delta: \text{Ext}^*_{BP_*}(BP_*/(p)) \to \text{Ext}^{*+1}_{BP_*}(BP_*),$$

with $\delta(h_{i+1}) = \theta_i$. Applying $\delta$ to 6.4.7 gives

$$(6.4.8) \quad \theta_i \theta_i^p = \theta_{i+1} \theta_0^p \in \text{Ext}_{(BP_*,BP_*)}(BP_*,BP_*).$$

We can now prove the theorem by induction on $i$, using 4.4.22 to start the induction. We have for $i > 0$

$$d_{2p-1}(\theta_{i+1}) \theta_0^p = d_{2p-1}(\theta_{i+1} \theta_0^p)$$

$$= d_{2p-1}(\theta_i \theta_i^p)$$

$$= d_{2p-1}(\theta_i \theta_i^p)$$

$$= h_0 \theta_i^{p-1} \theta_1^p \mod \ker \theta_0^{a_{i-1}}$$

$$= h_0 (\theta_i^{p-1} \theta_1^{p-1})^p$$

$$= h_0 (\theta_i^{p-1} \theta_0^{p-1})^p$$

$$= h_0 \theta_i^p \theta_0^p$$

so

$$d_{2p-1}(\theta_{i+1}) \equiv h_0 \theta_i^p \mod \ker \theta_0^{a_i}. \quad \square$$
We now turn to the proof of 6.4.4. We map $\text{Ext}(BP_\ast) \to \text{Ext}(v^{-1}BP_\ast/I_n)$ with $n = p - 1$. By 6.1.1 this group is isomorphic to $\text{Ext}_{S_n}(K(n)_\ast, K(n)_\ast)$, which is essentially the cohomology of the profinite group $S_n$ by 6.2.4. By 6.2.12 $S_n$ has a subgroup of order $p$ since the field $K$ obtained by adjoining $p\text{th}$ roots of unity to $Q_p$ has degree $p - 1$. We will show that the elements of 6.4.4 have nontrivial images under the resulting map to the cohomology of $\mathbb{Z}/(p)$. In other words, we will consider the composite

$$BP_\ast(BP) \to \Sigma(n) \to S(n) \otimes \mathbb{F}_p^n \to C,$$

where $C$ is the linear dual of the group ring $\mathbb{F}_p^n[\mathbb{Z}/(p)]$.

6.4.9. Lemma. Let $C$ be as above. As a Hopf algebra

$$C = \mathbb{F}_p^n[t]/(t^p - t) \quad \text{with} \quad \Delta t = t \otimes 1 + 1 \otimes t.$$

Proof. As a Hopf algebra we have $\mathbb{F}_p^n[\mathbb{Z}/(p)] = \mathbb{F}_p^n[u]/(u^p - 1)$ with $\Delta u = u \otimes u$, where $u$ corresponds to a generator of the group $\mathbb{Z}/(p)$. We define an element $t \in C$ by its Kronecker pairing $\langle u_i, t \rangle = i$. Since the product in $C$ is dual to the coproduct in the group algebra, we have

$$\langle u_i, t^k \rangle = \langle \Delta(u_i), t \otimes t^{k-1} \rangle = \langle u_i, t \rangle \langle u_i, t^{k-1} \rangle$$

so by induction on $k$

(6.4.10) $$\langle u_i, t^k \rangle = i^k.$$

We also have $\langle u_i, 1 \rangle = 1$.

We show that $\{1, t, t^2, \ldots, t^{p-1}\}$ is a basis for $C$ by relating it to the dual basis of the group algebra. Define $x_j \in C$ by

$$x_j = \sum_{0 < k < p} (jt)^k$$

for $0 < j < p$ and $x_0 = 1 + \sum_{0 < j < p} x_j$. Then

$$\langle u_i, x_j \rangle = \left\langle u_i, \sum_{0 < k < p} (jt)^k \right\rangle = \sum_{0 < k < p} j^k i^k$$

and

$$\langle u_i, x_0 \rangle = \left\langle u_i, 1 + \sum_{0 < j < p} x_j \right\rangle = \left\{ \begin{array}{ll} 1 & \text{if } i = 0 \\ 0 & \text{ otherwise} \end{array} \right.$$
To proceed with the proof of 6.4.4: we now show that under the epimorphism
\[ f: \Sigma(n) \otimes K(n) \to C \quad \text{(where } n = p - 1), \quad f(t_1) \neq 0. \]
From the proof of 6.2.3, \( t_1 \) can be regarded as a continuous function from \( S_n \) to \( \mathbb{F}_{p^n} \). It follows then that the nontriviality of \( f(t_1) \) is equivalent to the nonvanishing of the function \( t_1 \) on the nontrivial element of order \( p \) in \( S_n \). Suppose \( x \in S_{p-1} \) is such an element. We can write
\[ x = 1 + \sum_{i>0} e_i S^i \]
with \( e_i \in W(\mathbb{F}_{p^n}) \) and \( e_i^{p^n} = e_i \). Recalling that \( S^{p-1} = p \), we compute
\[ 1 = x^p \equiv 1 + pe_1 S + (e_1 S)^p \mod (S)^{1+p} \]
and
\[ (e_1 S)^p \equiv e_1^{(p^p-1)/(p-1)} S \mod (S)^{1+p} \]
so it follows that
\[ e_1 + e_1^{(p^p-1)/(p-1)} \equiv 0 \mod (p). \]
[Remember that \( t_1(x) \) is the mod \((p)\) reduction of \( e_1 \).] Clearly, one solution to this equation is \( e_1 \equiv 0 \mod (p) \) and hence \( e_1 = 0 \). We exclude this possibility by showing that it implies that \( x = 1 \). Suppose inductively that \( e_i = 0 \) for \( i < k \). Then \( x \equiv 1 + e_k S^k \mod (S^{k+1}) \) and \( x^p \equiv 1 + pe_k S^k \mod (S^{k+p}) \) so \( e_k \equiv 0 \mod (p) \). Since \( e_k^{p^n} - e_k = 0 \), this implies \( e_k = 0 \).
Hence, \( f \) is a map of Hopf algebras, \( f(t_1) \) primitive, so \( f(t_1) = ct \) where \( c \in \mathbb{F}_{p^n} \) is nonzero. Now recall that
\[ \text{Ext}_C(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) = H^*(\mathbb{Z}/(p); \mathbb{F}_{p^n}) = E(h) \otimes P(b), \]
where \( E() \) and \( P() \) denote exterior and polynomial algebras over \( \mathbb{F}_{p^n} \), respectively, \( h = [t] \in H^1 \), and
\[ b = \sum_{0 < j < p} \frac{1}{p} \binom{p}{j} [j^p] \in H^2. \]
Let \( f^* \) denote the composition
\[ \text{Ext}(BP_* \to \text{Ext}(v^{-1}_n BP_* / I_n) \xrightarrow{\sim} \text{Ext}_{\Sigma(n)}(K(n)_*, K(n)) \to \text{Ext}_C(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \xrightarrow{\sim} H^*(\mathbb{Z}/(p); \mathbb{F}_{p^n}). \]
Then it follows that \( f^*(h_0) = -ch \) and \( f^*(h_i) = -c^i b \) and 6.4.4 is proved.

Note that the scalar \( c \) must satisfy \( 1 + c^{(p^p-1)/(p-1)} = 0 \) Since \( c^{p^p-1} - 1 = 1 \), the equation is equivalent to \( 1 + c^{(p^p-1)/(p-1)} = 0 \). It follows that \( c = w^{(p-1)/2} \) for some generator \( w \) of \( \mathbb{F}^{\times}_{p-1} \), so \( c \) is not contained in any proper subfield of \( \mathbb{F}_{p^n-1} \). Hence tensoring with this field is essential to the construction of the detecting map \( f \).

Now we examine the corresponding situation in the Adams spectral sequence. The relations used to prove 6.4.1 (apart from the assertion of nontriviality) are also valid here, but the machinery used to prove 6.4.4 is, of course, not available. Indeed the monomials vanish in some cases. The following result was first proved by May [1].

6.4.11. Proposition. For \( p = 3 \), \( h_0 b_3^2 = 0 \) in \( \text{Ext}_{A_*}(\mathbb{Z}/(3), \mathbb{Z}/(3)) \); i.e., \( b_2 \) cannot support the expected nontrivial differential.
PROOF. We use a certain Massey product identity (A1.4.6) and very simple facts about \( \text{Ext}_{A_{q}}(\mathbb{Z}/(3), \mathbb{Z}/(3)) \) to show \( h_{0}b_{21}^{2} = 0 \). We have
\[
 h_{0}b_{21}^{2} = -h_{0}(h_{1}, h_{1})b_{1} = -(h_{0}, h_{1})h_{1}b_{1}.
\]
By (6.4.14) \( h_{1}b_{1} = h_{2}b_{0} \), so
\[
 h_{0}b_{21}^{2} = -(h_{0}, h_{1})h_{2}b_{0} = -(h_{1}, h_{0})h_{2}b_{0} = -(h_{0}, h_{1})h_{2}b_{0}.
\]
The element \((h_{0}, h_{1}, h_{2})\) is represented in the cobar construction by \( \xi_{1}^{3}\xi_{2} + \xi_{3}^{2}\xi_{1} \), which is the coboundary of \( \xi_{3} \), so \( h_{0}b_{21}^{2} = 0 \).

The case of \( b_{2} \) at \( p = 3 \) is rather peculiar. One can show in the Adams–Novikov spectral sequence that \( d_{5}(\beta_{7}) = \pm \alpha_{3}\beta_{7}^{3/3} \). (This follows from the facts that \( d_{5}(\beta_{4}) = \pm \alpha_{1}\beta_{4}^{2}\beta_{3/3} \), \( \beta_{4}^{2} = \pm \beta_{1}\beta_{7} \), \( \beta_{4}\beta_{3/3} = \pm \beta_{1}\beta_{6/3} \), and \( \beta_{3/3} = \pm \beta_{1}^{2}\beta_{6/3} \). We leave the details to the reader.) Hence \( \beta_{7} \) is a permanent cycle mapping to \( b_{2} \). The elements \( \beta_{7} \) and \( \alpha_{3}\beta_{3/3} = \pm \alpha_{1}\beta_{3/3} \) correspond to Adams elements in filtrations 8 and 10 which are linked by a differential. We do not know the fate of the \( b_{i} \) at \( p = 3 \) for \( i > 2 \).

To prove 6.4.2 we will need two lemmas.

6.4.12. Lemma. For \( p \geq 3 \)

(i) \( \text{Ext}^{2}\mathbf{q}^{-2}(BP_{*}) \) is generated by the \( [(i + 3)/2] \) elements \( \beta_{a_{i}/p^{i+3-2j}} \), where \( j = 1, 2, \ldots, [(i + 3)/2] \), \( a_{i,j} = (p^{i+2}+p^{i+3-2j})/(p+1) \), and \( [(i + 3)/2] \) is the largest integer \( \leq (i + 3)/2 \). Each of these elements has order \( p \).

(ii) Each of these elements except \( \beta_{p^{i+1}/p^{i+1}} \) reduces to zero in
\[
\text{Ext}^{2}\mathbf{q}^{-2}(BP_{*}/I_{3}).
\]

6.4.13. Lemma. For \( p \geq 5 \), any element of \( \text{Ext}^{2}\mathbf{q}^{-2}(BP_{*}) \) (for \( i \geq 0 \)) which maps to \( b_{i+1} \) in the Adams \( E_{2} \)-term supports a nontrivial differential \( d_{2p-1} \).

We have seen above that 6.4.13 is false for \( p = 3 \).

Theorem 6.4.2 follows immediately from 6.4.13 because a permanent cycle in the Adams spectral sequence of filtration 2 must correspond to one in the Adams–Novikov spectral sequence of filtration \( \leq 2 \). By sparseness (4.4.2) the Novikov filtration must also be 2, but 6.4.13 says that no element in \( \text{Ext}^{2}(BP_{*}) \) mapping to \( b_{i} \) for \( i \geq 1 \) can be a permanent cycle.

PROOF of 6.4.12. Part (i) can be read off from the description of \( \text{Ext}^{2}(BP_{*}) \) given in 5.4.5.

To prove (ii) we recall the definition of the elements in question. We have short exact sequences of \( BP_{*}(BP) \)-comodules
\[
(6.4.14) \quad 0 \to BP_{*} \to BP_{*} \xrightarrow{p} BP_{*}/(p) \to 0.
\]
\[
(6.4.15) \quad 0 \to BP_{*}/(p) \xrightarrow{v_{1}^{i+3-2j}} BP_{*}/(p) \to BP_{*}/(p,v_{1}^{i+3-2j}) \to 0.
\]
Let \( \delta_{0} \) and \( \delta_{1} \), denote the respective connecting homomorphisms. Then we have \( v_{2}^{a_{i,j}} \in \text{Ext}_{0}^{0}(BP_{*}BP_{*}/(p,v_{1}^{i+3-2j})) \) and \( \beta_{a_{i+1}/p^{i+3-2j}} = \delta_{0}\delta_{1}(v_{2}^{a_{i,j}}) \). The element \( \beta_{p^{i+1}/p^{i+1}} \) the above element for \( j = 1 \) can be shown to be \( b_{i+1} \) as follows.

The right unit formula 4.3.21 gives
\[
(6.4.16) \quad \eta_{R}(v_{2}) = v_{2} + v_{1}t_{1}^{p} - v_{1}^{p}t_{1} \quad \text{mod } (p),
\]
\[
\delta_{1}(v_{2}^{p^{i+1}}) = t_{1}^{p^{i+2}} - v_{1}^{p^{i+2}-p^{i+1}}t_{1}^{p^{i+1}}.
\]
and \( \delta_0(t_1^{i+2}) = b_{i+1} \). Moreover 6.4.16 implies that in \( \text{Ext}(BP_*/(p)) \),
\[ v_1^{p^i} t_1^{p^{i+1}} \cong v_1^{p^{i+1}} t_1^p, \text{ so } v_1^{p^{i+2}} - t_1^{p^{i+1}} \cong v_1^{p^{i+2}} - t_1^{p+1}. \]
This element is the mod \((p)\) reduction of \( p^{-i-2} \delta_0(t_1^{i+2}) \) and is therefore in \( \ker \delta_0 \).
Hence \( \delta_0 \delta_1(v_2^{i+2}) = \delta_0(t_1^{p+2}) = b_{i+1} \).
This definition of \( \beta_{i+1}/p^{i+1} \) differs from that of 5.4.5, where for \( i > 0 \) it is defined to be \( \delta_0 \delta_1(v_2^{i+2} - v_1^{i+2} - t_1^{p+1})p^{-i} \).

In principle one can compute this element explicitly in the cobar complex (A1.2.11) and reduce mod \( I_3 \), but that would be very messy. A much easier method can be devised using Yoneda’s interpretation of elements in Ext groups as equivalence classes of exact sequences (see, for example, Chapter IV of Hilton and Stammbach [1]) as in 5.1.20(b). Consider the following diagram.

(6.4.17)

\[
\begin{array}{cccccccc}
0 & \longrightarrow & BP_* & \overset{p}{\longrightarrow} & BP_*/(p) & \overset{p_2}{\longrightarrow} & BP_*/(p, v_1^{i+3-2j}) & \longrightarrow 0 \\
\downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & & \\
0 & \longrightarrow & BP_*/(p, v_1, v_2) & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & BP_*/(p, v_1^{i+3-2j}) & \longrightarrow 0.
\end{array}
\]

The top row is obtained by splicing 6.4.14 and 6.4.15 and it corresponds to an element in \( \text{Ext}^2(BP_*/(p, v_1^{i+3-2j}), BP_*) \). Composing this element with
\[ v_2^{a_{i,j}} \in \text{Ext}^0(BP_*/(p, v_1^{i+3-2j})) \]
gives \( \beta_{a_{i,j}/p^{i+3-2j}} \).

We let \( p_1 \) be the standard surjection. It follows from Yoneda’s result that if we choose \( BP_* \) BP-comodules \( M_1 \) and \( M_2 \), and comodule maps \( p_2 \) and \( p_3 \) such that the diagram commutes and the bottom row is exact, then the latter will determine the element of
\[ \text{Ext}^2_{BP, BP}(BP_*/(p, v_1^{i+3-2j}), BP_*/(p, v_1, v_2)) \]
which, when composed with \( v_2^{a_{i,j}} \), will give the mod \( I_3 \) reduction of \( \beta_{a_{i,j}/p^{i+3-2j}} \). We choose \( M_1 = BP_*/(p^2, pv_1, v_1^2, pv_2) \) and \( M_2 = BP_*/(p, v_1^{2+p^{i+3-2j}}) \) and let \( p_2 \) and \( p_3 \) be the standard surjections. It is easy to check that \( M_1 \) and \( M_2 \) are comodules over \( BP_*(BP) \), i.e., that the corresponding ideals in \( BP_* \) are invariant. (The ideal used to define \( M_1 \) is simply \( I_2^2 + I_4 I_3 \).) Moreover, the resulting diagram has the desired properties.

The resulting bottom row of 6.4.17 is the splice of the two following short exact sequences.

(6.4.18) \[ 0 \rightarrow BP_*/(p, v_1, v_2) \overset{p}{\longrightarrow} BP_*/(p^2, pv_1, pv_2, v_1^2) \rightarrow BP_*/(p, v_1^2) \rightarrow 0, \]
(6.4.19) \[ 0 \rightarrow BP_*/(p, v_1^2) \overset{v_1^{i+3-2j}}{\longrightarrow} BP_*/(p, v_1^{2+p^{i+3-2j}}) \rightarrow BP_*/(p, v_1^{i+3-2j}) \rightarrow 0. \]

Let \( \delta_0', \delta_1' \) denote the corresponding connecting homomorphisms. The elements we are interested in then are \( \delta_0' \delta_1'(v_2^{a_{i,j}}). \)

To compute \( \delta_1'(v_2^{a_{i,j}}) \) we use the formula \( d(v_2^n) = (v_2 + v_1 t_1^p - v_1^{p+1})n - v_2^n \), implied by 6.4.16, in the cobar construction for \( BP_*/(p, v_1^{2+p^{i+3-2j}}) \). Recall that
\[ a_{i,j} = (p^{i+2} + p^{i+3-2j})/(p+1) \quad 1 \leq j \leq [(i+3)/2]. \]
Hence $a_{i,j} = p^{i+3-2j} \mod (p^{i+4-2j})$ and $d(v_2^{a_{i,j}}) = v_2^{b_{i,j}} p^{i+3-2j} [t_1^{p^{i+4-2j}}]$, so

$$\delta_t^1(v_2^{a_{i,j}}) = v_2^{b_{i,j}} [t_1^{p^{i+3-2j}}],$$

where $b_{i,j} = a_{i,j} - p^{i+3-2j} = (p^{i+2} - p^{i+4-2j})/(p + 1)$.

For $j = 1$, $b_{i,1} = 0$ and

$$\delta_t^0(\delta_t^1(v_2^{a_{i,j}})) = -\sum_{0<k<p} \frac{1}{p} \left(\frac{b}{p}\right) [t_1^{k}\left[t_1^{(p-k)p}\right]] = -b_{i+1}.$$

For $j > 1$, $b_{i,j}$ is divisible by $p$ and $d(v_2^{b_{i,j}}) \equiv 0 \mod (p^2, pv_1, v_2^2)$ and

$$v_2^{b_{i,j}} d(p^{i+4-2j}) \equiv 0 \mod (pv_2),$$

so $\delta_t^1(v_2^{a_{i,j}}) \in \text{Ext}^1(BP_*/(p, v_2^2))$ pull back in 6.4.17 to an element of

$$\text{Ext}^1(BP_*/(p^2, pv_1, pv_2, v_2^2)) \quad \text{and} \quad \delta_t^0(\delta_t^1(v_2^{a_{i,j}})) = 0,$$

completing the proof. \qed

**Proof of 6.4.13.** Any element of $\text{Ext}^{2,qp^{i+2}}(BP_*)$ can be written uniquely as $c b_{i+1} + x$ where $x$ is in the subgroup generated by the elements $\beta_{a_{i,j},p^{i+3-2j}}$ for $j > 1$. In 5.4.6, we showed that $x$ maps to zero in the classical Adams $E_2$-term. Hence it suffices to show that no such $x$ can have the property

$$d_{2p-1}(x) = d_{2p-1}(b_{i+1}).$$

By 5.5.2 for $p \geq 5$ there is an 8-cell spectrum $V(2) = M(p, v_1, v_2)$ with $BP_*(V(2)) = BP_*/(p, v_1, v_2)$, and a map $f: S^0 \to V(2)$ inducing a surjection in $BP$ homology. $f$ also induces the standard map

$$f_*: \text{Ext}(BP_*) \to \text{Ext}(BP_*/I_3).$$

Lemma 6.4.12 asserts that $f_*(\beta_{a_{i,j},p^{i+3-2j}}) = 0$ for $j > 1$, so $f_*(d_{2p-1}(x)) = 0$ where $x$ is as above. However, 6.4.1 and the proof of 6.4.4 show that

$$g_*(d_{2p-1}(b_{i+1})) \neq 0,$$

where $g_*$ is induced by the obvious map

$$g: BP_* \to v_2^{-1}BP_*/I_{p-1}.$$  

Since $g$ factors through $BP_*/I_3$, this shows that $f_*(d_{2p-1}(b_{i+1})) \neq 0$, completing the proof. \qed

5. **The Spectra $T(m)$**

In this section we construct certain spectra $T(m)$ and study the corresponding chromatic spectral sequence. $T(m)$ satisfies

$$BP_*(T(m)) = BP_*[t_1, t_2, \ldots, t_m] \subset BP_*(BP_*)$$

as a comodule algebra. These are used in Chapter 7 in a computation of the Adams–Novikov $E_2$-term. We will see there that the Adams–Novikov spectral sequence for $T(m)$ is easy to compute through a range of dimensions that grows rapidly with $m$, and here we will show that its chromatic spectral sequence is very regular.

To construct the $T(m)$ recall that $BU = \Omega SU$ by Bott periodicity, so we have maps $\Omega SU(k) \to BU$ for each $k$. Let $X(k)$ be the Thom spectrum of
the corresponding vector bundle over \( \Omega SU(k) \). An easy calculation shows that \( H_*(X(k)) = \mathbb{Z}[b_1, b_2, \ldots, b_{k-1}] \subset H_*(MU) \). Our first result is

6.5.1. Splitting Theorem. For any prime \( p \), \( X(k)_{(p)} \) is equivalent to a wedge of suspensions of \( T(m) \) with \( m \) chosen so that \( p^m \leq k < p^{m+1} \), and \( BP_* T(m) = BP_*[t_1, \ldots, t_m] \subset BP_*(BP) \). Moreover \( T(m) \) is a homotopy associative commutative ring spectrum.

From this we get a diagram

\[
S^0_{(p)} = T(0) \to T(1) \to T(2) \to \cdots \to BP.
\]

In Ravenel [8, §3] we show that after \( p \)-adic completion there are no essential maps from \( T(i) \) to \( T(j) \) if \( i > j \) or from \( BP \) to \( T(i) \).

This theorem is an analog of 4.1.12, which says that \( MU_{(p)} \) splits into a wedge of suspensions of \( BP \), as is its proof. We start with the following generalization of 4.1.1.

6.5.2. Definition. Let \( E \) be an associative commutative ring spectrum. A complex orientation of degree \( k \) for \( E \) is a class \( x_E \in \tilde{E}^2(\mathbb{C}P^k) \) whose restriction to \( \tilde{E}^2(\mathbb{C}P^1) \cong \pi_0(E) \) is 1.

A complex orientation as in 4.1.1 is of degree \( k \) for all \( k > 0 \). This notion is relevant in view of

6.5.3. Lemma. \( X(k) \) admits a complex orientation of degree \( k \).

Proof. \( X(k) \) is a commutative associative ring spectrum (up to homotopy) because \( \Omega SU(k) \) is a double loop space. The standard map \( \mathbb{C}P^{k-1} \to BU \) lifts to \( \Omega SU(k) \). Thomifying gives a stable map \( \mathbb{C}P^k \to X(k) \) with the desired properties.

\( X(k) \) plays the role of \( MU \) in the theory of spectra with orientation of degree \( k \). The generalizations of lemmas 4.1.4, 4.1.7, 4.1.8, and 4.1.13 are straightforward. We have

6.5.4. Proposition. Let \( E \) be an associative commutative ring spectrum with a complex orientation \( x_E \in \tilde{E}^2(\mathbb{C}P^k) \) of degree \( k \).

(a) \( E^*(\mathbb{C}P^k) = \pi_*(E)[x_E]/(x_E^{k+1}) \).

(b) \( E^*(\mathbb{C}P^k \times \mathbb{C}P^k) = \pi_*(E)[x_E \otimes 1, 1 \otimes x_E]/(x_E^{k+1} \otimes 1, 1 \otimes x_E^{k+1}) \).

(c) For \( 0 < i < k \) the map \( t: \mathbb{C}P^i \times \mathbb{C}P^{k-i} \to \mathbb{C}P^k \) induces a formal group law \( k \)-chunk; i.e., the element \( t^*(x_E) \) in the truncated power series ring \( \pi_*(E)[x_E \otimes 1, 1 \otimes x_E]/(x_E \otimes 1, 1 \otimes x_E)^{k+1} \) has properties analogous to an formal group law (A2.1.1).

(d) \( E_*(X(k)) = \pi_*(E)[b^E_1, \ldots, b^E_{k-1}] \) where \( b^E_i \in E_2(X(k)) \) is defined as in 4.1.7.

(e) With notation as in 4.1.8, in \( (E \wedge X(k))^2(\mathbb{C}P^k) \) we have

\[
\hat{x}_{X(k)} = \sum_{0 \leq i \leq k-1} b^E_i x_E^{i+1} \quad \text{where} \quad b_0 = 1.
\]

This power series will be denoted by \( g_E(\hat{x}_E) \).

(f) There is a one-to-one correspondence between degree \( k \) orientations of \( E \) and multiplicative maps \( X(k) \to E \) as in 4.1.13. \( \square \)
We do not have a generalization of 4.1.15, i.e., a convenient way of detecting maps \( X(k) \to X(k) \), but we can get by without it. By 6.5.4(f) a multiplicative map \( g: X(k)_{(p)} \to X(k)_{(p)} \) is determined by a polynomial \( f(x) = \sum_{0 \leq i < k - 1} f_i x^{i+1} \) with \( f_0 = 1 \) and \( f_i \in \pi_{2i}(X(k)_{(p)}) \). In this range of dimensions \( \pi_* X(k) \) is isomorphic to \( \pi_* (MU) \), so we can take \( f(x) \) to be the truncated form of the power series of \( A2.1.23 \). Then the calculations of 4.1.12 show that \( g \) induces an idempotent in ordinary or \( BP_* \)-homology. In the absence of 4.1.15 it does not follow that \( g \) itself is idempotent. Nevertheless we can define

\[
T(m) = \lim_g X(k)_{(p)},
\]

i.e., \( T(m) \) is the mapping telescope of \( g \). Then we can compose the map \( X(k)_{(p)} \to T(m) \) with various self-maps of \( X(k)_{(p)} \) to construct the desired splitting, thereby proving 6.5.1.

Now we consider the chromatic spectral sequence for \( T(m) \). Using the change-of-rings isomorphism 6.1.1, the input needed for the machinery of Section 5.1 is \( \text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m))) \) where \( K(n)_*(T(m)) = K(n)_*[t_1, \ldots, t_m] \). Using notation as in 6.3.7, let \( \Sigma(n, m + 1) = \Sigma(n)/(t_1, \ldots, t_m) \). Then we have

6.5.5. Theorem. With notation as above we have

\[
\text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m))) = K(n)_*[u_{n+1}, \ldots, u_{m+n}] \otimes_{K(n)} \text{Ext}_{\Sigma(n,m+1)}(K(n)_*, K(n)),
\]

where \( \text{dim } u_j = \text{dim } v_j \). Moreover \( u_j \) maps to \( v_j \) under the map to \( \text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(BP)) = B(n)_* \) (6.1.11) induced by \( T(m) \to BP \). In other words its image in \( K(n)_*(BP) \) coincides with that of \( \eta_R(v_j) \) in \( BP_*(BP) \) under the map \( BP_*(BP) \to K(n)_*(BP) \).

Applying 6.3.7 gives

6.5.6. Corollary. If \( n < m + 2 \) and \( n < 2(p - 1)(m + 1)/p \) then

\[
\text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m))) = K(n)_*[u_{n+1}, \ldots, u_{m+n}] \otimes E(h_{k,j}: m + 1 \leq k \leq m + n, \ j \in \mathbb{Z}/(n)).
\]

The proof of 6.5.5. The images of \( \eta_R(v_{n+j}) \) (for \( 1 \leq j \leq m \)) in \( K(n)_*(T(m)) \) are primitive and give the \( u_{n+j} \). The image of \( BP_*(T(m)) \to BP_*(BP) \to \Sigma(n) \) is the subalgebra generated by \( \{ t_n : n \leq m \} \). The result follows by a routine argument.

Now we will use the chromatic spectral sequence to compute \( \text{Ext}^s(BP_*(T(m))) \) for \( s = 0 \) and 1. We assume \( m > 0 \) since \( T(0) = S^0 \), which was considered in 5.2.1 and 5.2.6. By 6.5.5 and 6.5.6 we have

\[
\text{Ext}_{\Sigma(0)}(K(0)_*, K(0)_*(T(m))) = \mathbb{Q}[u_1, \ldots, u_m] \quad \text{and} \quad \text{Ext}_{\Sigma(1)}(K(1)_*, K(1)_*(T(m))) = K(1)_*[u_2, \ldots, u_{m+1}] \otimes E(h_{m+1,0}).
\]

The short exact sequence

\[
0 \to M^0 \otimes BP_*(T(m)) \xrightarrow{i} M^1 \otimes BP_*(T(m)) \xrightarrow{p} M^1 \otimes BP_*(T(m)) \to 0
\]
induces a six-term exact sequence of Ext groups with connecting homomorphism $\delta$. For $j \leq m$, $\eta_R(v_j) \in BP_*(T(m)) \subset BP_*(BP)$, so if $u$ is any monomial in these elements then $\delta u/p^j = 0$ for all $i > 0$ and $\Ext^0(M^1 \otimes BP_*(T(m)))$ has a corresponding summand isomorphic to $\mathbb{Q}/\mathbb{Z}(p)$. Hence in the chromatic spectral sequence, $E_{1,0}^1$ has a summand isomorphic to $\mathbb{Z}(p)[u_1, \ldots, u_m] \otimes \mathbb{Q}/\mathbb{Z}(p)$, which is precisely the image of $d_1: E_{1,0}^1 \to E_{1,0}^1$, giving

6.5.9. **Proposition.**

$$\Ext^0(BP_*(T(m))) = \mathbb{Z}(p)[u_1, \ldots, u_m].$$

Next we need to consider the divisibility of $u_{m+1}/p \in \Ext^0(M^1 \otimes BP_*(T(m)))$. Note that $\eta_R(v_{m+1})$ is not in $BP_*(T(m))$ but $\eta_R(v_{m+1}) = pt_{m+1}$ (where $v_{m+1}$ is Hazewinkel’s generator given by A2.2.1) is, so we call this element $u_{m+1}$. It follows that in the cobar complex $C(BP_*(T(m)))$ (A1.2.11) $d(u_{m+1}) = pt_{m+1}$ and

$$d(u_{m+1}) \equiv ptu_{m+1} + u^2 \left( \frac{t+2}{2} \right) u_{m+1}^2 \mod (p^2 t),$$

where the second term is nonzero only when $p = 2$ and $t$ is even. Thus the situation is similar to that for $m = 0$ where we have $v_1 = u_1$. Recall that in that case the presence of the second term caused $Ext^1$ to behave differently at $p = 2$. We will show that this does not happen for $m \geq 1$ and we have

6.5.11. **Theorem.** For $m \geq 1$ and all primes $p$

$$\Ext^1(BP_*(T(m))) = \Ext^0(BP_*(T(m))) \otimes \{u_{m+1}/pt: t > 0\}.$$

**Proof.** For $p > 2$ the result follows from 6.5.10 as in 5.2.6. Now recall the situation for $m = 0$, $p = 2$. For $t = 2$, 6.5.10 gives $d(v_2) = 4(v_1 t_1 + t_1^2)$ and we have $d(4v_1^{-1}v_2) \equiv 4(v_2 t_1 + t_1^2) \mod (8)$, so we get a cocycle $(v_2^2 + 4v_1^{-1}v_2)/8$. The analogous cocycle for $m \geq 1$ would be something like

$$(u_{m+2}^2 + 4v_1^{-1}u_{m+2})/q$$

where $u_{m+2}$ is related somehow to $v_{m+2}$. However, the relevant terms in $\eta_R(v_{m+2}) \mod (2)$ are $v_1 t_{m+1}^2 + 2v_2 t_{m+1}$, which does not bear the resemblance to 6.5.10 for $m \geq 1$ that it does for $m = 0$. In other words $u_{m+1}^{-1}t_{m+1}^2$ is not cohomologous mod (2) to $u_{m+1}^{-1}t_{m+1}$, so the calculation for $p = 2$ can proceed as it does for $p > 2$.}

Our last result is useful for computing the Adams–Novikov $E_2$-term for $T(m)$ by the method used in Section 4.4.

6.5.12. **Theorem.** For $t < 2(p^{2m+2} - 1)$

$$\Ext(BP_*(T(m))/I_{m+1}) = \mathbb{Z}/(p)[u_{m+1}, u_{m+2}, \ldots, u_{2m+1}] \otimes E(h_{i,j}) \otimes P(b_{i,j})$$

with $i \geq m + 1$, $i + j \leq 2m + 2$, $h_{i,j} \in \Ext^{1,2p^j(p^j-1)}$ and $b_{i,j} \in \Ext^{2,2p^j+1(p^j-1)}$.

6.5.13. **Example.** For $m = 1$ we have

$$\Ext(BP_*(T(1))/I_2) = \mathbb{Z}/(p)[u_2, u_3] \otimes E(h_{2,0}, h_{2,1}, h_{2,2}, h_{3,0}, h_{3,1})$$

$$\otimes P(b_{2,0}, b_{2,1}, b_{3,0})$$

in 6.5.1 for $t \leq 2(p^4 - 1)$.
Proof of 6.5.12. By a routine change-of-rings argument (explained in Sec-
tion 7.1) the Ext in question is the cohomology of $C_\Gamma(BP_*/I_{m+1})$ (A1.2.11) where $\Gamma = BP_*(BP)/(t_1, \ldots, t_m)$. Then from 4.3.15 and 4.3.20 we can deduce that $v_i$ and $t_i$ are primitive for $m + 1 \leq i \leq 2m + 1$. $h_{i,j}$ corresponds to $t_i^{p^j}$ and $b_{i,j}$ to $-\sum_{0 < k < p} p^{-1}(p) t_i^{kp^j} l_i^{(p-k)p^j}$. The result follows by routine calculation. \qed
Bibliography

Adams, J. F.
Adams, J. F. and Priddy, S. B.
Adams, J. F. and Margolis, H. R.
Adams, J. F.
Adams, J. F.
Adams, J. F., Gunawardena, J. H., and Miller, H. R.
Aikawa, T.
Anderson, D. W. and Davis, D. W.
Anderson, D. W. and Hodckin, L.
Ando, M., Hopkins, M. J., and Strickland, N. P.
Araki, S.  

Aubry, M.  

Baas, N. A.  

Bahri, A. P. and Mahowald, M. E.  

Barratt, M. G., Mahowald, M. E., and Tangora, M. C.  

Barratt, M. G., Jones, J. D. S., and Mahowald, M. E.  

Behrens, M. and Pemmaraju, S.  

Bendersky, M., Curtis, E. B., and Miller, H. R.  

Bendersky, M.  

Bendersky, M., Curtis, E. B., and Ravenel, D. C.  

Bott, R.  


Bousfield, A. K. and Kan, D. M.  


Bousfield, A. K. and Kan, D. M.  

Bousfield, A. K. and Curtis, E. B.  

Browder, W.  

Brown, E. H. and Peterson, F. P.  

Brown, E. H.  

Brown, E. H. and Gitler, S.  
Bruner, R. R., May, J. P., McClure, J. E., and Steinberger, M.

Bruner, R. R.

Carlsson, G.

Cartan, H. and Eilenberg, S.

Cartier, P.

Cassels, J. W. S. and Fröhlich, A.

Cohen, F., Moore, J. C., and Neisendorfer, J.

Cohen, R. L. and Goerss, P.

Cohen, R. L.

Conner, P. E. and Floyd, E. E.

Conner, P. E. and Smith, L.

Conner, P. E.

Curtis, E. B.

Davis, D. M. and Mahowald, M. E.

Davis, D. M.

Davis, D. M. and Mahowald, M. E.

Davis, D. M.
Devinatz, E. S. and Hopkins, M. J. 

Devinatz, E. S., Hopkins, M. J., and Smith, J. H. 

Dieudonné, J. 

Dold, A. and Thom, R. 

Eckmann, B. 

Eilenberg, S. and Moore, J. C. 

Eilenberg, S. and MacLane, S. 

Elmendorf, A. D., Kriz, I., Mandell, M. A., and May, J. P. 

Floyd, E. E. 

Fröhlich, A. 

Giambalvo, V. and Pengelley, D. J. 

Giambalvo, V. 

Gorbounov, V. and Symonds, P. 

Gray, B. W. 

Gunawardena, J. H. C. 

Harper, J. R. and Miller, H. R. 

Hatcher, A. 

Hazewinkel, M. 


Henn, H.-W.

Higgins, P. J.

Hilton, P. J. and Stambach, U.

Hirzebruch, F.

Hopf, H.

Hopkins, M. J., Kuhn, N. J., and Ravenel, D. C.

Hopkins, M. J. and Mahowald, M. A.

Hopkins, M. J. and Smith, J. H.

Hurewicz, W.

James, I. M.


Johnson, D. C., Miller, H. R., Wilson, W. S., and Zahler, R. S.

Johnson, D. C. and Wilson, W. S.


Johnson, D. C. and Yosimura, Z.

Johnson, D. C. and Wilson, W. S.
Kahn, D. S.

Kahn, D. S. and Priddy, S. B.

Kambe, T., Matsunaga, H., and Toda, H.

Kochman, S. O.

Kriz, I.

Kuhn, N. J.

Landweber, P. S.

Landweber, P. S., Ravenel, D. C., and Stong, R. E.

Lang, S.

Lannes, J.

Lannes, J. and Schwartz, L.

Lazard, M.
Li, H. H. and Singer, W. M.

Lin, W. H.

Lin, W. H., Davis, D. M., Mahowald, M. E., and Adams, J. F.

Lin, W. H.

Liulevicius, A. L.

Liulevicius, A.


Lubin, J.


Lubin, J. and Tate, J.

Mac Lane, S.

MacLane, S.

Mahowald, M. E.

Mahowald, M.


Mahowald, M. E.

Mahowald, M. and Milgram, R. J.

Mahowald, M. E.


Mahowald, M. E. and Tangora, M. C.

Mahowald, M. E.


Mahowald, M. E. and Tangora, M. C.

Mahowald, M. E.

Mäkinen, J.

Margolis, H. R., Priddy, S. B., and Tangora, M. C.

Massey, W. S.


Maund, C. R. F.


May, J. P.


Milgram, R. J.


Miller, H. R., Ravenel, D. C., and Wilson, W. S.

Miller, H. R.

Miller, H. R. and Wilson, W. S.
Miller, H. R. and Ravenel, D. C.

Miller, H. R.

Miller, H. R. and Wilkerson, C.

Miller, H. R.

Milnor, J. W.

Milnor, J. W.

Milnor, J. W. and Stasheff, J. D.

Mironov, O. K.

Mischenko, A. S.
[1] Appendix 1 in Novikov [1].

Moore, J. C. and Peterson, F. P.

Morava, J.

Moreira, M. R. F.

Moss, R. M. F.
Mahowald, M., Ravenel, D., and Shick, P.

Mumford, D.

Nakamura, O.

Nassau, C.

Neisendorfer, J. A.

Times, N. Y.

Nishida, G.

Novikov, S. P.

Oka, S.

Oka, S. and Toda, H.

Oka, S. and Shimomura, K.

Palmieri, J. H.
BIBLIOGRAPHY 387

Pengelley, D. J.

Priddy, S. B.

Quillen, D. G.

Quillen, D.

Ravenel, D. C.

Ravenel, D. C. and Wilson, W. S.

Ravenel, D. C.

Rezk, C.

Richter, W.

Schwartz, L.

Science
Serre, J.-P.

Shimada, N. and Yagita, N.

Shimada, N. and Iwai, A.

Shimada, N. and Yamanoshita, T.

Shimomura, K.

Shimomura, K. and Wang, X.

Siegel, C. L.

Silverman, J. H.

Singer, W. M.

Smith, L.

Spanier, E. H.
Steenrod, N. E. and Epstein, D. B. A.

Stong, R. E.
[2] Determination of \( H^*(BO(k, \ldots, \infty), \mathbb{Z}_2) \) and \( H^*(BU(k, \ldots, \infty), \mathbb{Z}_2) \), Trans. Amer. Math. Soc. 107 (1963), 526–544.

Strickland, N. P.

Sullivan, D. P.

Sweedler, M. E.

Switzer, R. M.

Tangora, M. C.

Thom, R.

Thomas, E. and Zahler, R. S.

Toda, H.

tom Dieck, T.
Wang, J. S. P.

Welcher, P. J.

Wellington, R. J.

Wilson, W. S.

Würgler, U.

Yagita, N.

Yosimura, Z.

Zachariou, A.

Zahler, R.