ON $\beta$-ELEMENTS IN THE ADAMS-NOVIKOV SPECTRAL SEQUENCE
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Dedicated to Professor Takao Matumoto
on his sixtieth birthday

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Abstract. In this paper we detect invariants in the comodule consisting of $\beta$-elements over the Hopf algebroid $(A(m+1), G(m+1))$ defined in [Rav02], and we show that some related Ext groups vanish below a certain dimension. The result obtained here will be extensively used in [NR] to extend the range of our knowledge for $\pi_*(T(m))$ obtained in [Rav02].

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1. Introduction

In this paper we describe some tools needed in the method of infinite descent, which is an approach to finding the $E_2$-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres. It is the subject of [Rav86, Chapter 7], [Rav04, Chapter 7] and [Rav02].

We begin by reviewing some notation. Fix a prime $p$. Recall the Brown-Peterson spectrum $BP$. Its homotopy groups and those of $BP \wedge BP$ are known to be

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polynomial algebras
\[ \pi_*(BP) = \mathbb{Z}_p[v_1, v_2, \ldots] \quad \text{and} \quad \text{BP}_*(BP) = \text{BP}_*[t_1, t_2, \ldots]. \]

In [Rav86, Chapter 6] the second author constructed intermediate spectra

\[ S^0_{(p)} = T(0) \rightarrow T(1) \rightarrow T(2) \rightarrow T(3) \rightarrow \cdots \rightarrow \text{BP} \]

with \( T(m) \) is equivalent to \( \text{BP} \) below the dimension of \( v_{m+1} \). This range of dimensions grows exponentially with \( m \). \( T(m) \) is a summand of \( p \)-localization of the Thom spectrum of the stable vector bundle induced by the map \( \Omega SU(p^m) \rightarrow \omega SU = \text{BU} \).

In [Rav02] we constructed truncated versions \( T(m)_{(j)} \) for \( j \geq 0 \) with

\[ T(m) = T(m)_{(0)} \rightarrow T(m)_{(1)} \rightarrow T(m)_{(2)} \rightarrow \cdots \rightarrow T(m + 1) \]

These spectra satisfy

\[ \text{BP}_*(T(m)) = \pi_*(BP)[t_1, \ldots, t_m] \quad \text{and} \quad \text{BP}_*(T(m)_{(j)}) = \text{BP}_*(T(m)) \{ t_{m+1} : 0 \leq \ell < p^j \} \]

Thus \( T(m)_{(j)} \) has \( p^j \) ‘cells,’ each of which is a copy of \( T(m) \).

For each \( m \geq 0 \) we define a Hopf algebroid

\[ \Gamma(m + 1) = (\text{BP}_*, \text{BP}_*(BP)/(t_1, t_2, \ldots, t_m)) = \text{BP}_*[t_{m+1}, t_{m+2}, \ldots] \]

with structure maps inherited from \( \text{BP}_*(BP) \), which is \( \Gamma(1) \) by definition. Let

\[ A = \text{BP}_*, \quad A(m) = \mathbb{Z}_p[v_1, \ldots, v_m] \quad \text{and} \quad G(m + 1) = A(m + 1)[t_{m+1}] \]

with \( t_{m+1} \) primitive. Then there is a Hopf algebroid extension

\[ (A(m + 1), G(m + 1)) \rightarrow (A, \Gamma(m + 1)) \rightarrow (A, \Gamma(m + 2)). \]

In order to avoid excessive subscripts, we will use the notation

\[ \hat{v}_i = v_{m+i}, \quad \text{and} \quad \hat{t}_i = t_{m+i}. \]

We will use the usual notation without hats when \( m = 0 \). We will use the notation

\[ \hat{v}_i = v_{m+i}, \quad \hat{t}_i = t_{m+i}, \quad \hat{\beta}_{i/e_1, e_0} = \frac{\hat{v}_i}{p^o v_i^{e_1}} \quad \text{and} \quad \hat{\beta}_{i/e_1} = \frac{\hat{v}_i}{p^o v_i^{e_1}}. \]

We will also use the notations \( \hat{\beta}_{i/e_1} = \hat{\beta}_{i/e_1, 1} \) and \( \hat{\beta}_{i/e_1} = \hat{\beta}_{i/e_1, 1} \) for short. We will use the usual notation without hats when \( m = 0 \).

Given a Hopf algebroid \( (B, \Gamma) \) and a \( \Gamma \)-comodule \( M \), we will abbreviate \( \text{Ext}_\Gamma(B, M) \) by \( \text{Ext}_\Gamma(M) \) and \( \text{Ext}_\Gamma(B) \) by \( \text{Ext}_\Gamma \). With this in mind, there are change-of-rings isomorphisms

\[ \text{Ext}_{\text{BP}_*(BP)}(\text{BP}_*(T(m))) = \text{Ext}_{\Gamma(m+1)} \]

and

\[ \text{Ext}_{\text{BP}_*(BP)}(\text{BP}_*(T(m)_{(j)})) = \text{Ext}_{\Gamma(m+1)} \left( T^{(j)}_m \right) \]

where \( T^{(j)}_m = A \{ \hat{t}_i : 0 \leq \ell < p^j \} \).
Very briefly, the method of infinite descent involves determining the groups

\[ \text{Ext}_{\Gamma(m+1)}^{1}(T^{(j)}_{m}) \quad \text{and} \quad \pi_{s}((T(m))_{j}) \]

by downward induction on \( m \) and \( j \).

To begin with, we know that

\[ \text{Ext}_{\Gamma(m+1)}^{0}\left( A \{ \tau_{m+1}^{\ell} : 0 \leq \ell < p^{j} \} \right) = A(m) \{ \tilde{\tau}_{i}^{j} : 0 \leq \ell < p^{j} \} \].

To proceed further, we make use of a short exact sequence of \( \Gamma(m+1) \)-comodules

\[ 0 \rightarrow BP_{\ast} \xrightarrow{\iota_{0}} D_{m+1}^{0} \xrightarrow{\rho_{0}} E_{m+1}^{1} \rightarrow 0, \tag{1.2} \]

where \( D_{m+1}^{0} \) is weak injective (meaning that its higher Ext groups vanish) with \( \iota_{0} \) inducing an isomorphism in \( \text{Ext}^{0} \). It has the form

\[ D_{m+1}^{0} = A(m)[\hat{\lambda}_{1}, \hat{\lambda}_{2}, \ldots] \subset Q \otimes BP_{\ast} \]

with

\[ \hat{\lambda}_{i} = p^{-1}\hat{\tau}_{i} + \cdots. \]

Thus we have an explicit description of \( E_{m+1}^{1} \), which is a certain subcomodule of the chromatic module \( N^{1} = BP_{\ast}/(p^{\infty}) \).

It follows that the connecting homomorphism \( \delta_{0} \) associated with (1.2) is an isomorphism

\[ \text{Ext}_{\Gamma(m+1)}^{s}(E_{m+1}^{1}) \xrightarrow{\cong} \text{Ext}_{\Gamma(m+1)}^{s+1}(T^{(j)}_{m}) \]

for each \( s \geq 0 \). The determination of this group for \( s = 0 \) will be the subject of [Nak]. In this paper we will limit our attention to the case \( s > 0 \).

Unfortunately there is no way to embed \( E_{m+1}^{1} \) in a weak injective comodule in a way that induces an isomorphism in \( \text{Ext}^{0} \) as in (1.2). (This is explained in [NR, Remark7.4].) Instead we will study the Cartan-Eilenberg spectral sequence for \( \text{Ext}_{\Gamma(m+1)}^{s}(E_{m+1}^{1} \otimes T^{(j)}_{m}) \) associated with the extension (1.1). Its \( E_{2} \)-term is

\[ \tilde{E}_{2}^{s,t}(T^{(j)}_{m}) = \text{Ext}_{\Gamma(m+1)}^{s}(\text{Ext}_{\Gamma(m+2)}^{t}(T^{(j)}_{m} \otimes E_{m+1}^{1})) \]

\[ = \text{Ext}_{\Gamma(m+1)}^{s}(T^{(j)}_{m} \otimes \text{Ext}_{\Gamma(m+2)}^{t}(E_{m+1}^{1})) \] \( \tag{1.3} \)

where \( T^{(j)}_{m} = A(m+1) \{ \tilde{\tau}_{i}^{j} : 0 \leq \ell < p^{j} \} \)

and differentials \( \tilde{d}_{r} : \tilde{E}_{2}^{s,t} \rightarrow \tilde{E}_{2}^{s+r,t-r+1} \). Note that \( T^{(j)}_{m} = A \otimes_{\Gamma(m+1)} T^{(j)}_{m} \). We use the tilde to distinguish this spectral sequence from the resolution spectral sequence. We did not use this notation in [Rav02].

The short exact sequence of \( \Gamma(m+1) \)-comodules (1.2) is also a one of \( \Gamma(m+2) \)-comodules, and \( D_{m+1}^{0} \) is also weak injective over \( \Gamma(m+2) \) (this was proved in [Rav02, Lemma 2.2]), but this time the map \( \iota_{0} \) does not induce an isomorphism in \( \text{Ext}^{0} \). However, the connecting homomorphism

\[ \delta_{0} : \text{Ext}_{\Gamma(m+2)}^{t}(E_{m+1}^{1} \otimes T^{(j)}_{m}) \rightarrow \text{Ext}_{\Gamma(m+2)}^{t+1}(T^{(j)}_{m}) \]
is an isomorphism of $G(m + 1)$-comodules for $t > 0$. Note that over $\Gamma(m + 2)$, $T^{(j)}_m$ is a direct sum of $p^j$ suspended copies of $A$, so the isomorphism above is the tensor product with $T^{(j)}_m$ with

$$\delta_0 : \text{Ext}^t_{\Gamma(m+2)}(E^{1}_{m+1}) \to \text{Ext}^{t+1}_{\Gamma(m+2)}.$$

We will abbreviate the group on the right by $U^{t+1}_{m+1}$. Its structure up to dimension $(p^2 + p)|\hat{v}_2|$ was determined in [NR, Theorem 7.10]. It is $p$-torsion for all $t \geq 0$ and $v_1$-torsion for $t > 0$. Moreover, it is shown that each $U^{t}_{m+1}$ for $t \geq 2$ is a certain suspension of $U^{2}_{m+1}$ below dimension $p|\hat{v}_3|$.

Let $\mathcal{E}^{1}_{m+1} = \text{Ext}^0_{\Gamma(m+2)}(E^{1}_{m+1})$. For $j = 0$, the Cartan-Eilenberg $E_2$-term of (1.3) is

$$\tilde{E}_2^{s,t}(T^{(0)}_m) = \begin{cases} 
\text{Ext}^{s}_{G(m+1)}(\mathcal{E}^{1}_{m+1}) & \text{for } t = 0 \\
\text{Ext}^{s}_{G(m+1)}(U^{t+1}_{m+1}) & \text{for } t \geq 1.
\end{cases}$$

While it is impossible to embed the $\Gamma(m+1)$-comodule $E^{1}_{m+1}$ into a weak injective by a map inducing an isomorphism in $\text{Ext}^0$, it is possible to do this for the $G(m+1)$-comodule $\mathcal{E}^{1}_{m+1}$. In Theorem 2.4 below we will show that there is a pullback diagram of $G(m+1)$-comodules

$$0 \to \mathcal{E}^{1}_{m+1} \to W_{m+1} \to B_{m+1} \to 0$$

where $W_{m+1}$ is weak injective, $\iota_1$ induces an isomorphism in $\text{Ext}^0$, and $B_{m+1}$ is the $A(m+1)$-submodule of $\mathcal{E}^{1}_{m+1}/(v_1^\infty)$ generated by

$$\left\{ \frac{\hat{v}_2^i}{ip^i} : i > 0 \right\}.$$

The object of this paper is to study $B_{m+1}$ and related Ext groups. Since the $i$th element above is $\hat{\beta}^{(i)}_1$, the elements of $B_{m+1}$ are the beta elements of the title.

In [NR] we construct a variant of the Cartan-Eilenberg spectral sequence converging to $\text{Ext}^0_{\Gamma(m+1)}(T^{(j)}_m)$. Its $\tilde{E}_1$-term has the following chart:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$s$</th>
<th>$\text{Ext}^0$</th>
<th>$\text{Ext}^1$</th>
<th>$\text{Ext}^2$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>$U^3$</td>
<td>$U^3$</td>
<td>$U^3$</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$U^2$</td>
<td>$U^2$</td>
<td>$U^2$</td>
<td>...</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$\mathcal{D}$</td>
<td>$W$</td>
<td>$B$</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$s = 0$</td>
<td>$\text{Ext}^0$</td>
<td>$\text{Ext}^0$</td>
<td>$\text{Ext}^1$</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$s = 1$</td>
<td>$\text{Ext}^0$</td>
<td>$\text{Ext}^0$</td>
<td>$\text{Ext}^1$</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$s = 2$</td>
<td>$\text{Ext}^0$</td>
<td>$\text{Ext}^0$</td>
<td>$\text{Ext}^1$</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$s = 3$</td>
<td>$\text{Ext}^0$</td>
<td>$\text{Ext}^0$</td>
<td>$\text{Ext}^1$</td>
<td>...</td>
</tr>
</tbody>
</table>

where all Ext groups are over $G(m+1)$ and the tensor product signs and subscripts (equal to $m+1$) on $U^{t+1}$, $\mathcal{D}$, $W$ and $B$ have been omitted to save space.
Tensoring (1.4) with $T_{m}^{(j)}$, we also have the following diagram:

$$
\begin{array}{ccccccccc}
& \vdots & \vdots & \vdots & \vdots \\
\begin{array}{cccc}
\text{Ext}^0(T_{m}^{(j)} U^3) & \text{Ext}^1(T_{m}^{(j)} U^3) & \text{Ext}^2(T_{m}^{(j)} U^3) & \cdots \\
\text{Ext}^0(T_{m}^{(j)} U^2) & \text{Ext}^1(T_{m}^{(j)} U^2) & \text{Ext}^2(T_{m}^{(j)} U^2) & \cdots \\
\text{Ext}^0(T_{m}^{(j)} W) & \text{Ext}^0(T_{m}^{(j)} B) & \text{Ext}^1(T_{m}^{(j)} B) & \cdots \\
0 & 0 & 0 & \cdots \\
\end{array}
\end{array}
\begin{array}{cccc}
t=0 & t=1 & t=2 & \cdots \\
s=0 & s=1 & s=2 & s=3
\end{array}
$$

The construction of $B_{m+1}$ will be given in §2. After introducing our basic methodology in §3, we determine the groups $\text{Ext}^0(T_{m}^{(j)} \otimes B_{m+1})$ for the cases $j = 0, j = 1$ and $j > 1$ in the next three sections. Here

$$T_{m}^{(j)} = A(m + 1) \{ t_{m+1}^\ell : 0 \leq \ell < p^j \}.$$ 

In §7 we determine the higher Ext groups for $j = 1$ in a range of dimensions. Our calculations require some results about binomial coefficients and Quillen operations that are collected in Appendices A and B respectively.

2. The construction of $B_{m+1}$

**Proposition 2.1.** A 4-term exact sequence of $G(m + 1)$-comodules. The short exact sequence (1.2) gives a 4-term exact sequence

$$
\begin{array}{cccccccc}
0 & \longrightarrow & T_{m+1}^0 & \longrightarrow & A(m)[p^{-1}\hat{v}_1] & \longrightarrow & E_{m+1}^1 & \longrightarrow & U_{m+1}^1 & \longrightarrow & 0.
\end{array}
$$

Let

$$V_{m+1} = A(m)[p^{-1}\hat{v}_1]/A(m + 1)$$

$$= A(m + 1) \left\{ \frac{\hat{\lambda}_i}{p^i} : i > 0 \right\} \subset BP_*/(p^\infty).$$

There is a short exact sequence of $G(m + 1)$-comodules

$$
\begin{array}{cccccccc}
0 & \longrightarrow & V_{m+1} & \longrightarrow & E_{m+1}^1 & \longrightarrow & T_{m+1}^1 & \longrightarrow & 0
\end{array}
$$

which is not split.

**Proof.** The comodule $D_{m+1}^0$ was described explicitly in [Rav02, Theorem 3.9]. It has the form

$$D_{m+1}^0 = A(m)[\hat{\lambda}_1, \ldots] \subset p^{-1}BP_*$$
with

\[
\lambda_i = \begin{cases} 
\frac{\hat{v}_1}{p} & \text{for } i = 1 \\
\frac{\hat{v}_2}{p} + \frac{\hat{v}_1 v_{p^2}}{p^2} + \frac{(p^{p-1}-1)v_1 \hat{v}_1^p}{p^{p+1}} & \text{for } i = 2 \\
\frac{\hat{v}_i}{p} + \ldots & \text{for } i > 2
\end{cases}
\]

and

\[
\eta_R(\lambda_i) = \begin{cases} 
\hat{\lambda}_1 + \hat{t}_1 & \text{for } i = 1 \\
\hat{\lambda}_2 + \hat{t}_2 + (p^{p-1}-1)v_1 \sum_{0<j<p} p^{-1}(p^j) \hat{\lambda}_1^{p^j} & \text{for } i = 2 \\
\hat{\lambda}_i + \hat{t}_i + \ldots & \text{for } i > 2
\end{cases}
\]

It follows that \(\text{Ext}_0^{\Gamma(m+2)}(D_{m+1}^0) = A(m)[\hat{\lambda}_1]\) as claimed.

In order to understand the relation between \(E_{m+1}^1\) and \(U_{m+1}^1\), consider the following diagram of \(\Gamma(m+2)\)-comodules with exact rows.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & BP_* & \rightarrow & D_{m+1}^0 & \rightarrow & E_{m+1}^1 & \rightarrow & 0 \\
& & & & & & & & \\
0 & \rightarrow & BP_* & \rightarrow & p^{-1}BP_* & \rightarrow & BP_*/(p^\infty) & \rightarrow & 0 \\
& & & & & & & & \\
0 & \rightarrow & BP_* & \rightarrow & D_{m+2}^0 & \rightarrow & E_{m+2}^1 & \rightarrow & 0
\end{array}
\]

The vertical maps are monomorphisms, and there is no obvious map either way between \(D_{m+1}^0\) and \(D_{m+2}^0\). The description of the \(U_{m+1}^1 = \text{Ext}_1^{\Gamma(m+2)}\) above is in terms of the connecting homomorphism for the bottom row. The element

\[
\frac{\hat{v}_2}{p^i} \in E_{m+2}^1
\]

is invariant and maps to the similarly named element in \(U_{m+1}^1\). To describe its image in terms of the cobar complex, we pull it back to \(\hat{v}_2/p^i \in D_{m+2}^0\) and compute its coboundary, which is

\[
d(\hat{v}_2/p^i) = ((\hat{v}_2 + p\hat{t}_2)^i - \hat{v}_2^i)/p^i = \hat{v}_2^{i-1}\hat{t}_2 + \ldots
\]

However, the element \(\hat{v}_2/p^i\) is not present in \(E_{m+1}^1\). To see this, consider the case \(i = 1\). In \(p^{-1}BP_*\), we have

\[
\hat{v}_2/p = \frac{-\hat{\lambda}_2 - \hat{v}_1 v_{p^2}}{p^2} + \frac{(1-p^{p-1})v_1 \hat{v}_1^p}{p^{p+1}}
\]

\[
= \frac{-\hat{\lambda}_2 - \hat{v}_1 v_{p^2}}{p^2} + \frac{(1-p^{p-1})v_1 \hat{v}_1^p}{p}
\]

\[
\not\in D_{m+1}^0 = A(m)[\hat{\lambda}_1, \hat{\lambda}_2, \ldots].
\]

Instead of \(\hat{v}_2/p\), consider the element \(\hat{\lambda}_2\) itself. Its image in \(E_{m+1}^1\) is invariant, so it defines a nontrivial element in \(E_{m+1}^1\). The computation of the image of \((p\hat{\lambda}_2)/p^i\) under the connecting homomorphism gives the same answer as before.
The right unit formula above implies that the short exact sequence does not split. □

**Definition 2.2.** Let $M$ be a graded torsion $G(m+1)$-comodule of finite type, and let $M_i$ have order $p^{a_i}$. Then the Poincaré series for $M$ is defined by

\[
g(M) = \sum a_i t^i.
\]

Given two such power series $f_1(t)$ and $f_2(t)$, the inequality $f_1(t) \leq f_2(t)$ means that each coefficient of $f_1(t)$ is dominated by the corresponding one in $f_2(t)$.

**Theorem 2.4. Construction of $B_{m+1}$.** Let $B_{m+1} \subset E_{m+1}^1/(v_1^\infty)$ be the sub-$A(m+1)$-module generated by the elements

\[
\hat{\beta}_{i/i} = \frac{\hat{v}_2^i}{ip_0^i}
\]

for all $i > 0$. It is a $G(m+1)$-subcomodule whose Poincaré series is

\[
g(B_{m+1}) = g_{m+1}(t) \sum_{k \geq 0} \frac{x^{p^{k+1}}(1 - y^k)}{(1 - x^{p^{k+1}})(1 - x_2^{p^k})},
\]

where

\[
y = t^{v_1}, \quad x = t^{\hat{v}_1}, \quad x_i = t^{\hat{v}_i} \quad \text{for } i > 1
\]

and

\[
g_{m+1}(t) = \prod_{1 \leq i \leq m+1} \frac{1}{1 - t^{v_1}}.
\]

Let $W_{m+1}$ be the pullback in the diagram (1.4). Then $W_{m+1}$ is a weak injective with $\text{Ext}^0_{G(m+1)}(W_{m+1}) = \text{Ext}^0_{G(m+1)}(E_{m+1})$, i.e., the map $E_{m+1} \to W_{m+1}$ induces an isomorphism in $\text{Ext}^0$.

**Proof.** To show that $B_{m+1}$ is a $G(m+1)$-subcomodule, note that

\[
\eta_R(\hat{v}_2) = \hat{v}_2 + v_1 \hat{v}_1^p - v_1^{p^0} \hat{t}_1 \mod p
\]

so

\[
\eta_R(\hat{v}_2^i) = (\hat{v}_2 + v_1 \hat{v}_1^p - v_1^{p^0} \hat{t}_1)^p \mod p^i
\]

and

\[
\eta_R(\hat{\beta}_{i/i}) \in B_{m+1} \otimes G(m+1).
\]

so $B_{m+1}$ is a $G(m+1)$-comodule.

For the Poincaré series, let $F_kB_{m+1} \subset B_{m+1}$ denote the submodule of exponent $p^k$ with $F_0B_{m+1} = \phi$. Then the Poincaré series of

\[
F_kB_{m+1}/F_{k-1}B_{m+1} = A(m+1)/I_1 \left\{ \hat{\beta}_{ip^{k-1}/ip^{k-1},p^k} : i > 0 \right\}
\]
is
\[ g(F_kB_{m+1}/F_{k-1}B_{m+1}) = g(A(m+1)/I_2) \sum_{i>0} x^i p^i \frac{1 - y^{i+1}}{1 - y} \]
\[ = g_{m+1}(t) \sum_{i>0} (x^i p^i - (x^i y) p^{i+1}) \]
\[ = g_{m+1}(t) \sum_{i>0} (x^{i+1} - x^{i+1}) \]
\[ = g_{m+1}(t) \left( \frac{x^{i+1}}{1 - x^p} - \frac{x^{i+1}}{1 - x^{p+1}} \right). \]

Summing these for all positive \( k \) gives the desired formula.

To show \( \text{Ext}^0_{\Gamma(m+1)}(W_{m+1}) \) is as claimed it is enough to show that the connecting homomorphism
\[ \text{Ext}^0_{\Gamma(m+1)}(B_{m+1}) \longrightarrow \text{Ext}^1_{\Gamma(m+1)}(E_{m+1}) \]
is monomorphic. Since the target group is in the Cartan-Eilenberg \( \tilde{E}_2 \)-term converging to \( \text{Ext}^1_{\Gamma(m+1)}(E_{m+1}) \), we have the composition
\[ \eta : \text{Ext}^0_{\Gamma(m+1)}(B_{m+1}) \longrightarrow \text{Ext}^1_{\Gamma(m+1)}(E_{m+1}) \xrightarrow{\delta t} \text{Ext}^2_{\Gamma(m+1)}. \]
So it is sufficient to show that \( \eta \) is monomorphic. Since \( B_{m+1} \) is in \( \text{Ext}^0_{\Gamma(m+2)}(N^2) \), we have the following diagram
\[
\begin{array}{ccc}
\text{Ext}^0_{\Gamma(m+1)}(M^1) & \longrightarrow & \text{Ext}^0_{\Gamma(m+1)}(N^2) \longrightarrow \text{Ext}^1_{\Gamma(m+1)}(N^1) \\
\downarrow v^{-1}_{1} \text{Ext}^1_{\Gamma(m+1)} & & \downarrow v^{-1} \text{Ext}^0_{\Gamma(m+1)}(B_{m+1}) \eta \longrightarrow \text{Ext}^2_{\Gamma(m+1)}
\end{array}
\]
The right equality holds because \( \text{Ext}^1_{\Gamma(m+1)}(M^0) = 0 \), and the top row is exact. Since \( \text{Ext}^0_{\Gamma(m+1)}(M^1) \) is the \( v^{-1}_{1} \text{Ext}^1_{\Gamma(m+1)}(N^1) \)-module generated by \( \tilde{v}^{-1}_{1}/ip \) the map \( \eta \) is monomorphic as desired.

The Poincaré series of \( W_{m+1} \) is given by
\[ g(W_{m+1}) = g(E_{m+1}^1) + g(B_{m+1}) = g(V_{m+1}) + g(U_{m+1}^1 + g(B_{m+1}) \]
\[ = g_{m+1}(t) \left( \frac{x}{1 - x} + \sum_{j \geq 0} \frac{x^j}{1 - x^j} + \sum_{j \geq 0} \frac{x^{j+1}(1 - y^{j+1})}{(1 - x^{j+1})(1 - x^{j+1})} \right) \]
\[ = g_{m+1}(t) \left( \frac{x}{1 - x} + \sum_{j \geq 0} \frac{x^{j+1}}{1 - x^{j+1}} \right) = g_{m+1}(t) \sum_{j \geq 0} \frac{x^{j+1}}{1 - x^{j+1}} \]
\[ = \frac{g(\text{Ext}^0_{\Gamma(m+1)})}{1 - \frac{x}{1 - x}} \quad \text{by} \ [\text{Rav02, Theorem 3.17}] \]
\[ = \frac{g\left( \text{Ext}^0_{\Gamma(m+1)}(W_{m+1}) \right)}{1 - x}. \]
This means that $W_{m+1}$ is weak injective by [Rav02, Theorem 2.6].

3. Basic methods for finding comodule primitives

From now on, all Ext groups are understood to be over $G(m+1)$.

**Definition 3.1.** [Rav04, Definition 7.1.8] A $G(m+1)$-comodule $M$ is called $j$-free if the comodule tensor product $T_m^{(j)} \otimes_{A(m+1)} M$ is weak injective, i.e.,

$$\text{Ext}^n(A(m+1), T_m^{(j)} \otimes_{A(m+1)} M) = 0$$

for $n > 0$. The elements of $\text{Ext}^0$ are called $j$-primitives.

We will often abbreviate $\text{Ext}(A(m+1), N)$ by $\text{Ext}(N)$ for short. We will see in Proposition 3.3 that it is enough to consider a certain subgroup $L_j(M)$ of $M$ to detect elements of $\text{Ext}^0(T_m^{(j)} \otimes M)$. Given a right $G(m+1)$-comodule $M$ and the structure map $\psi_M : M \to G(m+1) \otimes M$, define the Quillen operation $\hat{r}_j : M \to M$ ($i \geq 0$) by $\hat{r}_j(z) \otimes \hat{r}_j$. In this paper all comodules are right comodules. In most cases the structure map is determined by the right unit formula.

**Definition 3.2.** The group $L_j(M)$. Denote the subgroup $\bigcap_{n \geq p^i} \ker \hat{r}_n$ of $M$ by $L_j(M)$. By definition, we have a sequence of inclusions

$$L_0(M) \subset L_1(M) \subset \cdots \subset L_j(M) \subset \cdots$$

and $L_0(M) = \text{Ext}^0(M)$.

The following result allows us to identify $j$-primitives with $L_j(M)$.

**Proposition 3.3.** [Rav02, Lemma 1.12] Identification of the $j$-primitives with $L_j(m)$. For a $G(m+1)$-comodule $M$, the map

$$(c \otimes 1)\psi_M : L_j(M) \longrightarrow \text{Ext}^0(T_m^{(j)} \otimes M)$$

is an isomorphism between $A(m+1)$-modules, where $c$ is the conjugation map.

When we detect elements of $L_j(M)$, it is enough to consider elements killed by $\hat{r}_{p^j}$ ($j \geq 0$), as one sees by the following proposition.

**Proposition 3.4.** A property of Quillen operations. If the Quillen operation $\hat{r}_{p^j}$ on a $G(m+1)$-comodule $M$ is trivial, then all operations $\hat{r}_n$ for $p^j \leq n < p^{j+1}$ are trivial.

**Proof.** Since $\hat{r}_i \hat{r}_j = \binom{i+j}{j} \hat{r}_{i+j}$ [Nak, Lemma 3.1] we have a relation $\hat{r}_{n-p^j} \hat{r}_{p^j} = \binom{n}{p^j} \hat{r}_n$. Observing that the congruence $\binom{n}{p^j} \equiv s \mod (p)$ for $sp^j \leq n < (s+1)p^j$, $\binom{n}{p^j}$ is invertible in $\mathbb{Z}_{(p)}$ whenever $p^j \leq n < p^{j+1}$, and the result follows. \qed

In the following sections we will determine the structure of $L_0(B_{m+1})$ in Proposition 4.2 and 4.4 and $L_1(B_{m+1})$ in Proposition 5.1 and 5.4 in all dimensions, and $L_j(B_{m+1})$ ($j > 1$) in Theorem 6.1 below dimension $[\frac{n^{p^{j+1}}}{p^j}]$. Then we need a method for checking whether all $j$-primitives ($j > 1$) are listed or not.

The following lemma gives an explicit criterion the $j$-freeness of a comodule $M$. 


Lemma 3.5. A Poincaré series characterization of $j$-free comodules. For a graded torsion connective $G(m+1)$-comodule $M$ of finite type, we have an inequality

$$(3.6) \quad g(M)(1 - x^{p^i}) \leq g(L_j(M)) \quad \text{where } x = [\hat{v}_1]$$

with equality holding iff $M$ is $j$-free.

Proof. Let $I \subset A(m+1)$ be the maximal ideal. We have the inequality

$$g(T_m^{(j)} \otimes M) \leq g(\text{Ext}^0(T_m^{(j)} \otimes M)) \cdot g(G(m+1)/I)$$

by [Rav04] Theorem 7.1.34, where the equality holds iff $M$ is a weak injective. Observe that

$$g(T_m^{(j)} \otimes M) = g(M) \frac{1 - x^{p^i}}{1 - x},$$

$$g(G(m+1)/I) = \frac{1}{1 - x}$$

and

$$g(\text{Ext}^0(T_m^{(j)} \otimes M)) = g(L_j(M)).$$

□

Lemma 3.7. A Poincaré series formula for the first $\text{Ext}^1$ group. For a graded torsion connective $G(m+1)$-comodule $M$ of finite type, suppose

$$\frac{g(L_j(M))}{1 - x^{p^i}} - g(M) \equiv ct^d \mod t^{d+1}$$

Then the first nontrivial element in $\text{Ext}^1(T_m^{(j)} \otimes M)$ occurs in dimension $d$, and the order of the group $G = \text{Ext}^1(T_m^{(j)} \otimes M)$ is $p^i$.

Proof. Since the inequality of (3.6) is an equality below dimension $d$, $M$ is $j$-free in that range, so $\text{Ext}^1(T_m^{(j)} \otimes M)$ vanishes below dimension $d$. Each element $x \in G$ is represented by a short exact sequence of the form

$$0 \to T_m^{(j)} \otimes M \to M' \to \Sigma^d A(m+1) \to 0.$$ 

If $x$ has order $p^i$, then we get a diagram

$$0 \to T_m^{(j)} \otimes M \to M' \to \Sigma^d A(m+1) \to 0$$

Since $G$ is a finite abelian $p$-group, it is a direct sum of cyclic groups. We can do the above for each of its generators and assemble them into an extension

$$0 \to T_m^{(j)} \otimes M \to M'' \to \Sigma^d G \otimes \mathbb{Z}_{(p)} A(m+1) \to 0$$

with $\text{Ext}^0_{G(m+1)}(M'') = L_j(M)$ through dimension $d$ and $\text{Ext}^1_{G(m+1)}(M'') = 0$, so $M''$ is weak injective through dimension $d$. 
If $|G| = p^k$, then we have

$$
g(M''') = g\left(\Sigma^d G \otimes Z(p) A(m+1)\right)
= g(M) \left(\frac{1-x^p}{1-x}\right) + bt^d g_{m+1}(t)
$$

Since $M'''$ is weak injective through dimension $d$, we have

$$
g(M''') \equiv g\left(\text{Ext}^0_{G(m+1)}(M''')\right) \frac{1-x}{1-x} \mod t^{d+1}
= g(L_j(M)) \frac{1-x^p}{1-x}
= g(M) \left(\frac{1-x^p}{1-x}\right) + ct^d
$$

so $b = c$. \hfill \Box

4. 0-PRIMITIVES IN $B_{m+1}$

In this section we determine the structure of $\text{Ext}^0(B_{m+1})$, i.e., the primitives in $B_{m+1}$ in the usual sense. We treat the cases $m > 0$ and $m = 0$ separately. The latter is more complicated because $v_1$ is not invariant over $\Gamma(1)$. Recall that the $G(m+1)$-comodule structure of $B_{m+1}$ is given by the right unit map $\eta_R$.

**Lemma 4.1. An approximation of the right unit.** The right unit map $\eta_R : A(m+2)_* \to G(m+2)$ on the Hazewinkel generators are expressed by

$$
\eta_R(\hat{v}_1) = \hat{v}_1 + p\hat{t}_1, \\
\eta_R(\hat{v}_2) \equiv \hat{v}_2 + v_1\hat{t}_1 - v_1^p\hat{t}_1 \mod (p)
$$

where $\omega = p^m$.

**Proof.** These directly follow from [MRW] (1.1) and (1.3). \hfill \Box

For a given integer $n$, denote the exponent of a prime $p$ in the factorization of $n$ by $\nu_p(n)$ as usual. In particular, $\nu_p(0) = \infty$. When the integer is a binomial coefficient $\binom{n}{k}$, we will write $\nu_p(\binom{n}{k})$ instead of $\nu_p\left(\binom{n}{k}\right)$.

Let $\hat{h}_j$ be the 1-dimensional cohomology class of $\hat{t}_1^p$.

**Proposition 4.2. Structure of $\text{Ext}^0(B_{m+1})$ for $m > 0$.** For $m > 0$, $\text{Ext}^0(B_{m+1})$ is the $A(m)$-module generated by

$$
\left\{p^k\hat{v}_1^s\hat{t}_1^{k^p}\hat{t}_{i^t} : i > 0, s \geq 0, k \geq 0, 0 < t \leq p^k \text{ and } \nu_p(i) \leq \nu_p(s) \right\}.
$$

The first nontrivial element in $\text{Ext}^1(B_{m+1})$ is

$$
\hat{h}_0\hat{h}_1 \in \text{Ext}^{1,2(p+1)(p^\omega-1)}(B_{m+1}).
$$
Proof. We may put $s = ap^k$ and $i = bp^k$ with $p 
mid b$ and $a \geq 0$. Observe that

$$\psi\left( \frac{v_1^{ap^k} v_2^{bp^{k+1}}}{bp^{k+1} v_1^t} \right) = \frac{v_1^{ap^k} (v_2^{bp^k} + v_1^{bp^{k+1}} - v_1^{bp^{k+1} + t})}{bp^{k+1} v_1^t} \quad \text{since } p 
mid b$$

and so the exhibited elements are invariant. On the other hand, we have nontrivial Quillen operations

$$\hat{r}_1(\hat{v}_1^{ip^k / t}) = - \hat{v}_1 v_2^{p \hat{v}_1^{ip^{k-1}}} + s \cdot \hat{v}_1 v_2^{ip^{k}} \quad \text{if } \nu_p(s) < \nu_p(i)$$

and

$$\hat{r}_{p^{k+1}}(\hat{v}_1^{ip^k / t}) = \hat{v}_1 v_2^{p \hat{v}_1^{ip^{k+1}}} + \cdots \quad \text{if } t > p^k,$$

where the missing terms in the second expression involve lower powers of $\hat{v}_1$ in the numerator or smaller powers of $v_1$ in the denominator.

This means each element $p^k \hat{v}_1^{ip^k / t}$ with $\nu_p(s) < \nu_p(i)$ supports a nontrivial $\hat{r}_1$, the targets of which are linearly independent. Similarly, each such monomial with $t > p^k$ supports a nontrivial $\hat{r}_{p^{k+1}}$. It follows that no linear combination of such elements is invariant, so $\text{Ext}^1$ is as stated.

For the second statement, note that $\hat{h}_0$ and $\hat{\beta}_1$ are the first nontrivial elements in $\text{Ext}^1$ and $\text{Ext}^0(B_{m+1})$ respectively, so if their product is nontrivial, the claim follows. It is nontrivial because there is no $x \in B_{m+1}$ with $\hat{r}_1(x) = \beta_1$.

We now turn to the case $m = 0$.

**Lemma 4.3.** Right unit in $G(1)$. The right unit $\eta_R : A(1) \to G(1)$ on the chromatic fraction $\frac{1}{ipv_1^t}$ is

$$\eta_R\left( \frac{1}{ipv_1^t} \right) = \sum_{k \geq 0} \binom{t + k - 1}{k} \frac{(-1)^k}{ipv_1^{k-1} v_1^k}.$$  

Note that this sum is finite because a chromatic fraction is nontrivial only when its denominator is divisible by $p$.

**Proof.** Recall the expansion

$$\frac{1}{(x+y)^t} = (x+y)^{-t} = x^{-t} + y^{-t} = x^{-t} \sum_{k \geq 0} \binom{-t}{k} \frac{y^k}{x^k}$$

and the formula $\eta_R(v_1^t) = (v_1 + pt_1)^t$ by Lemma 4.1. \( \square \)

**Proposition 4.4.** Structure of $\text{Ext}^0(B_1)$. For $m = 0$, $\text{Ext}^0(B_1)$ is the $\mathbb{Z}_p$-module generated by

$$\left\{ p^k \hat{v}_1^{ip^k / t} : i > 0, k \geq 0, 0 \leq t \leq p^k \text{ and } \nu_p(i) \leq \nu_p(t) \right\}. $$
The first nontrivial element in $\text{Ext}^1(B_1)$ is

$$h_0 \beta_1 \in \text{Ext}^{1,2(p^2-1)}(B_{m+1})$$

Proof. When $i$ and $t$ are as stated, we may set $t = ap^k$ and $i = bp^k$ with $p | b$ and $a > 0$. Observe that

$$\eta R \left( \frac{v_2^{bp^k + k}}{bp^k + 1 v_1^{ap^k}} \right) = \left( v_2^{p^k} + v_1^{p^k} t_1^{p^k+1} - v_1^{k+1} t_1^{p^k} \right) bp^k \sum_{n \geq 0} \binom{ap^k + n - 1}{n} \frac{(-t_1)^n}{bp^k + 1 - n v_1^{ap^k} + n}.$$

For $n > 0$, the binomial coefficient is divisible by $p^{k+1-n}$ by Lemma A.3 below, so the expression simplifies to

$$\eta R \left( \frac{v_2^{bp^k}}{bp^k + 1 v_1^{ap^k}} \right) = \frac{\left( v_2^{p^k} + v_1^{p^k} t_1^{p^k+1} - v_1^{k+1} t_1^{p^k} \right) bp^k}{bp^k + 1 v_1^{ap^k}}$$

and $p^k \beta_{v_1^{p^k} / t}$ is invariant by an argument similar to that of Lemma 4.2. On the other hand if either of the conditions on $i$ and $t$ fails, we have nontrivial Quillen operations

$$r_1 \left( p^k \beta_{v_1^{p^k} / t} \right) = - \frac{v_2^{p^k}}{p^{1-k} v_1^{p^k}} - \frac{t}{i} \frac{v_2^{p^k}}{v_1^{p^k+1}} \text{ if } \nu_p(i) > \nu_p(t)$$

or

$$r_{p^k+1} \left( p^k \beta_{v_1^{p^k} / t} \right) = \frac{v_2^{p(i-1)p^k}}{p v_1^{p^k+1}} \text{ if } t > p^k.$$

The rest of the argument, including the identification of the first nontrivial element in $\text{Ext}^1(B_1)$, is the same as in the case $m > 0$. □

5. 1-PRIMITIVES IN $B_{m+1}$

In this section we determine the structure of $L_1(B_{m+1})$, which includes all elements of $\text{Ext}^0(B_{m+1})$ determined in the previous section. By observing that $\tilde{r}_1(\tilde{v}_i \beta_p') = \beta_p$ and $\tilde{r}_{p'}(\tilde{v}_i \beta_p') = 0$ for $j \geq 1$, the first element of the quotient $L_1(B_{m+1})/L_0(B_{m+1})$ is $\tilde{v}_1 \beta_p'$ for $m > 0$. In general, we have

**Proposition 5.1. Structure of $L_1(B_{m+1})$ for $m > 0$.** For $m > 0$, $L_1(B_{m+1})$ is isomorphic to the $A(m)$-module generated by $p^k \beta_{v_1^{p^k} / t}$, where $i > 0$, $s \geq 0$, $k \geq 0$ and $0 < t \leq p^k$, and the integers $i$ and $s$ satisfy the following condition: there is a non-negative integer $s$ such that $s \equiv 0, 1, \ldots p - 1 \mod (p^{n+1})$ and $\nu_p(i) < s + n + 1$.

Note that the description of $L_1(B_{m+1})$ differs from that of $L_0(B_{m+1})$ given in Proposition 4.2 only in the restriction on $i$ and $s$. In that case it was $\nu_p(i) \leq \nu_p(s)$. If $\nu_p(s) = n + 1$ (i.e., $s \equiv 0 \mod (p^{n+1})$), then an integer $i$ satisfying $\nu_p(i) \leq n + 1$ also satisfies $\nu_p(i) < n + p$. Hence we have $L_0(B_{m+1}) \subset L_1(B_{m+1})$ as desired.
Proof. In Proposition 4.2 we have already seen that \( p^k \beta_{ip^k/t} \) is invariant if \( 0 < t \leq p^k \). If follows that

\[
\tilde{r}_p(p^k \tilde{v}_1^s \beta_{ip^k/\nu}) = \tilde{r}_p(\tilde{v}_1^s) \cdot p^k \beta_{ip^k/\nu} = p^{p^s}(s)_{\nu}^{-p} \cdot \tilde{v}_2^{p^k}.
\]

Since we are dealing with 1-primitives, we can ignore the case \( \ell = 0 \). For \( \ell = 1 \), this is clearly trivial if \( s < p \). When \( s \geq p \), choose an integer \( n \) such that \( p^n | (s) \). By Lemma A.4 this means \( n = 0 \) unless \( s \) is \( p \)-adically close to an integer ranging from \( 0 \) to \( p - 1 \). Then \( \tilde{r}_p \) is trivial if \( \nu_p(i) < n + p \). We can show that all Quillen operations \( \tilde{r}_p \) for \( \ell > 1 \) are trivial under the same condition since

\[
\nu_p(p^k(s/p)) \leq \nu_p(p^{p^s}(s/p^s))
\]

which follows from

\[
q \nu_p(p^{p^s}(s/p^s)) = p^s + 1 + \alpha(s - p^s) - \alpha(s)
\]

by Lemma A.2

and

\[
q \left[ \nu_p(p^{p^s}(s/p^s)) - \nu_p(p^s(s/p)) \right] = p^s - p + \alpha(s - p^s) - \alpha(s - p)
\]

\[
\geq \alpha(p^s - p) + \alpha(s - p^s) - \alpha(s - p)
\]

\[
\geq 0.
\]

\[ \square \]

Note also that the condition on \( i \) and \( s \) in Proposition 5.1 is automatically satisfied whenever \( i < p^s \), which means that we may set \( n = 0 \). Since

\[
\tilde{r}_p(\tilde{v}_1^s) = p^p(s/p^s)^{s}_1^{-p}
\]

and \( p^p \) kills all of \( B_{m+1} \) below the dimension of \( \beta_{ip^k/\nu} \), \( \tilde{v}_1 \) is effectively invariant in this range, making \( B_{m+1} \) an \( A(m+1) \)-module.

Corollary 5.2. Poincaré series for \( L_1(B_{m+1}) \). For \( m > 0 \), the Poincaré series for \( L_1(B_{m+1}) \) below dimension \( p^p \tilde{v}_2 \) is

\[
g_{m+1}(t) = \sum_{k=0} g_{m+1}(t) \left( \frac{x^{p^{k+1}} - x^{p^k}}{1 - x^{p^k}} \right),
\]

and in the same range we have

\[
L_1(B_{m+1}) = A(m+1) \left\{ p^k \beta_{ip^k/t}: i > 0, k \geq 0, 0 < t \leq p^k \right\}.
\]

Proof. As is explained in the above, we may consider \( L_1(B_{m+1}) \) as an \( A(m+1) \)-module in that range. To determine the Poincaré series \( g(L_1(B_{m+1})) \), decompose \( L_1(B_{m+1}) \) into the following two direct summands:

(1) \( S_0 = A(m+1)/I_2 \left\{ \beta_i: i > 0 \right\} \)

(2) \( S_k = A(m+1)/I_2 \left\{ p^k \beta_{ip^k/t}: i > 0 \right\} \) for \( k > 0 \)
The Poincaré series for these sets are given by

\[ g(S_0) = g_{m+1}(t) \cdot (1 - y) \sum_{n \geq 0} \frac{y^{-1}}{1 - x_2^n} x_2^n \]

and

\[ g(S_k) = g_{m+1}(t) \cdot (1 - y) \sum_{n > 0} \frac{y^{-p^k}(1 - y^{p^k-1})}{1 - y} \frac{x_2^{n+k-1}}{1 - x_2^{p^n+k-1}} \]

which gives

\[ g(L_1(B_{m+1})) = \sum_{n \geq 0} \frac{x_2^n}{1 - x_2^n} + \sum_{0 < k \leq n} \frac{(y^{-p^k} - y^{-p^{k-1}})}{1 - x_2^n} x_2^n \]

\[ = \sum_{n \geq 0} (y^{-1} - 1) x_2^{p^n} \]

\[ = (y^{-1} - 1) x_2^{p^n} \sum_{n > 0} (y^{-p^n} - 1) \frac{x_2^n}{1 - x_2^n} \]

\[ = \sum_{n \geq 0} \frac{x_2^{p^n} (y^{-p^n} - 1)}{1 - x_2^n} \]

which is equal to (5.3).

Now we turn to the case \( m = 0 \), for which we make use of Lemma 4.3 again. Observing that \( \hat{r}_1(\beta'_p) = -\beta_{p/2} \) and \( \hat{r}_j(\beta'_p) = 0 \) for \( j \geq 1 \), the first element of the quotient \( L_1(B_{m+1})/L_0(B_{m+1}) \) is \( \beta'_p \). In general, we have

**Proposition 5.4. Structure of \( L_1(B_1) \).** For \( m = 0 \), \( L_1(B_1) \) is isomorphic to the \( \mathbf{Z}(p)_e \)-module generated by \( \beta^k \beta_{ip^{k+1}} \), where \( k \geq 0 \), \( i > 0 \) and \( 0 < t \leq p^k \) satisfying the following condition: there is a non-negative integer \( n \) such that \(-t = 0, 1, \ldots, p - 1 \) mod \((p^{n+1})\) and \( p^{n+1} \mid i \).

**Proof.** We have

\[ \psi \left( \frac{v_2^k}{ipv_1^1} \right) = \left( v_2^k + v_1^k \right) \left( -pt_1 \right)^r \sum_{r \geq 0} \binom{t + r - 1}{r} \frac{\left(-pt_1\right)^r}{ipv_1^{r+1}} \]

in which there are terms

\[ \frac{v_2^{(i-1)p^k}}{pv_1^{ip^{k+1}}} - \frac{v_1^{(i-1)p^k}}{pv_1^{ip^{k+1}}} \]

and \( (-p)^{r} \left( t + p^{e} - 1 \right) \frac{v_2^k}{ipv_1^{r+1}} \) for \( e \geq 0 \).

Since \( t \leq p^k \), the first and the second are trivial, which gives

\[ \hat{r}_{|p^k|} \left( p^k \beta_{ip^{k+1}} \right) = (-p)^{r} \left( t + p^{e} - 1 \right) \frac{v_2^k}{ipv_1^{r+1}} \]
Choose an integer \( n \) such that \( p^n \mid t^{p-1} \), which occurs if \(-t = 0, 1, \ldots, p - 1 \mod (p^{n+1})\) by Lemma A.4. Then \( \hat{\gamma}_p \) is trivial if \( p^{n+i} \). We can also observe that all the higher Quillen operations \( \hat{\gamma}_\ell \) (\( \ell \geq 1 \)) are trivial since \( \nu_p \left( p^\ell \left( t^{p-1} \right) \right) \leq \nu_p \left( p^\ell \left( t^{p-1} \right) \right) \) (see the proof of Proposition 5.1). \( \Box \)

**Corollary 5.5.** \( L_1(B_1) \) as an \( A(1) \)-module. For \( m = 0 \), we have

\[
L_1(B_1) = A(1) \left\{ p^k \beta_{t^k i^k} : i > 0, k \geq 0 \text{ and } 0 < t \leq p^k \right\}
\]

below dimension \( p^0 | v_2 \). The Poincaré series for \( L_1(B_1) \) in this range is the same as (5.3).

Applying Lemma 3.5 and 3.7 to the Poincaré series (5.3), we have the following result.

**Corollary 5.6.** 1-free range for \( B_{m+1} \). For \( m \geq 0 \), \( B_{m+1} \) is 1-free below dimension \( p(p + 1)|v_1 \), and the first element in \( \text{Ext}^1(T_m(1) \otimes B_{m+1}) \) is \( \hat{\beta}_{p/p} \).

Here we use the notation \( \hat{\beta}_{p/p} \) for its image under the map \((c \otimes 1) \psi_{B_{m+1}} \) (cf. (3.3)).

**Proof.** By comparing \( g(B_{m+1}) \) and \( g(L_1(B_{m+1})) \) and using Lemma 3.7, we see that the first nontrivial element of \( \text{Ext}^1(T_m(1) \otimes B_{m+1}) \) occurs in the indicated dimension, where the group has order \( p \). The fact that \( \hat{\beta}_{p/p} \) is nontrivial in \( \text{Ext}^1 \) follows by direct calculation. \( \Box \)

6. \( j \)-primitives in \( B_{m+1} \) for \( j > 1 \)

In this section we determine the structure of \( L_j(B_{m+1}) \) for \( j \geq 2 \) and \( m > 0 \) (See [Rav04] Lemma 7.3.1 for the \( m = 0 \) case). The first element of the quotient \( L_j(B_{m+1})/L_{j-1}(B_{m+1}) \) is \( \hat{\beta}_{p^{j-2+2}/p^{j-2+2}} \), which has nontrivial Quillen operation

\[
\hat{\gamma}_{p^{j-1}} \left( \hat{\beta}_{p^{j-2+2}/p^{j-2+2}} \right) = \hat{\beta}_1.
\]

In general, we have

**Theorem 6.1.** Structure of \( L_j(B_{m+1}) \) in low dimensions for \( j > 1 \).

(i) Below dimension \( p^{j+1}|v_2 | \), \( L_j(B_{m+1}) \) is the \( A(m+1) \)-module generated by

\[
\left\{ \hat{\beta}_{t/i} : 0 < t \leq \min(i, p^{j-1}) \right\} \cup \left\{ \hat{\beta}_{a+b/t} : p^{j-1} < t \leq p^j, a > 0 \text{ and } 0 \leq b < p^{j-1} \right\}.
\]

(ii) \( B_{m+1} \) is \( j \)-free below dimension \( |v_1^{p^{j+1}}| \).

(iii) The first element in \( \text{Ext}^1 \) is the \( p \)-fold Massey product

\[
\langle \hat{\beta}_{1+p^{j-1}/p^{j-1}}, \hat{\nu}_{1,j}, \ldots, \hat{\nu}_{1,j} \rangle_{p^{-1}}.
\]

For the basic properties of Massey products, we refer the reader to [Rav86, A1.4] or [Rav04, A1.4].
Proof. (i) The listed elements are the only \( j \)-primitives below dimensions \( p^{j+1} | \hat{\alpha}_2 | \) by Proposition B.3, and the first statement follows.

(ii) To show that \( B_{m+1} \) is \( j \)-free below the indicated dimension, we need to compute some Poincaré series. This will be a lengthy calculation.

Decompose \( L_j(B_{m+1}) \) into the following three direct summands:

\[
S_{0,1} = A(m + 1) \left\{ \beta_{\ell/t}^{p^{j-1}} : 0 < t \leq \ell < p^{j-1} \right\},
\]
\[
S_{0,2} = A(m + 1) \left\{ \beta_{\ell/t}^{p^{j-1}} : 0 < t \leq \ell < p^{j-1} \right\},
\]
\[
S_j = A(m + 1) \left\{ \beta_{ap^{j+b}/t}^{p^{j-1}} : p^{j-1} < t \leq p^j, a > 0 \text{ and } 0 \leq b < p^{j-1} \right\}.
\]

We will always work below the dimension of \( \hat{\beta}_{2p^{j}/p^j} \), which is \( |\hat{\alpha}_2^{p^{j+1}} | \). This means that in the description of \( S_j \) above, the only relevant value of \( a \) is 1.

Observe that

\[
S_{0,1} = \bigcup_{0 < k < j} A(m + 1)/I_2 \left\{ \frac{\hat{\alpha}_2^{k-1}}{p^k v_1^{p^{j-1} - \ell}} : 0 \leq \ell < p^{k-1}, 0 < i < p^{j-1} \right\},
\]

so

\[
g(S_{0,1}) = g(A(m + 1)/I_2) \sum_{0 < k < j} \sum_{0 < i < p^{j-1}} \frac{(1 - y^{ip^{k-1}})(x^p)^i}{1 - y} \]
\[
= g_{m+1}(t) \sum_{0 < k < j} \sum_{0 < i < p^{j-1}} (x^{ip^k} - x_2^{ip^{k-1}})
\]
\[
g(S_{0,1}) \quad \frac{g(S_{0,1})}{g_{m+1}(t)} = \sum_{0 < k < j} \left( \frac{x^{ip^k} - x^{ip^j}}{1 - x^{ip^k}} - \frac{x_2^{ip^{k-1}} - x_2^{ip^{j-1}}}{1 - x_2^{ip^{k-1}}} \right)
\]
\[
= \sum_{0 < k < j} \left( \frac{x^{ip^k} - x^{ip^j}}{1 - x^{ip^k}} - \frac{x_2^{ip^{k-1}} - x_2^{ip^{j-1}}}{1 - x_2^{ip^{k-1}}} \right)
\]

For \( S_{0,2} \), we have

\[
S_{0,2} = A(m + 1) \left\{ \frac{\hat{\alpha}_2}{ip v_1^{p^{j-1} - \ell}} : 0 \leq \ell < p^{j-1}, i \geq p^{j-1} \right\},
\]

which is the quotient of

\[
\bigcup_{k > 0} A(m + 1)/I_2 \left\{ \frac{\hat{\alpha}_2^{ip^{k-1}}}{p^k v_1^{p^{j-1} - \ell}} : 0 \leq \ell < p^{j-1}, i > 0 \right\}
\]
\[
\bigcup_{0 < k < j} A(m + 1)/I_2 \left\{ \frac{\hat{\alpha}_2^{ip^{k-1}}}{p^k v_1^{p^{j-1} - \ell}} : 0 \leq \ell < p^{j-1}, 0 < i < p^{j-1} \right\}.
\]
Hence the Poincaré series of $S_{0,2}$ is

$$g(S_{0,2}) = g(A(m + 1)/I_2) \cdot \frac{(1 - y^{p-1})y^{-p}}{1 - y}$$

$$\left( \sum_{k>0} \sum_{i>0} (x_2^{p-1})^i - \sum_{0<k<j} \sum_{0<i<p^{j-k}} (x_2^{p-1})^i \right)$$

$$g(S_{0,2}) \cdot g_m(t) = (y^{-p-1} - 1)$$

$$\left( \sum_{k>0} \frac{x_2^{p-1}}{1 - x_2^{p-1}} - \sum_{0<k<j} \frac{x_2^{p-1} - x_2^{p^{j-1}}}{1 - x_2^{p^{j-1}}} \right)$$

$$= (y^{-p-1} - 1)$$

$$\left( \sum_{k>j} \frac{x_2^{p-1}}{1 - x_2^{p-1}} + \sum_{0<k\leq j} \frac{x_2^{p^{j-1}}}{1 - x_2^{p^{j-1}}} \right)$$

$$\equiv (y^{-p-1} - 1)x_2^p + \sum_{0<k\leq j} \frac{x_2^p - x_2^{p^{j-1}}}{1 - x_2^{p^{j-1}}}$$

in our range of dimensions.

Adding these two gives

$$\frac{g(S_0 \cup S_{0,2})}{g_{m+1}(t)} = \frac{g(S_{0,1}) + g(S_{0,2})}{g_{m+1}(t)}$$

$$= \sum_{0<k<j} \left( \frac{x_2^k - x_2^p}{1 - x_2^k} - \frac{x_2^{p-1} - x_2^{p^{j-1}}}{1 - x_2^{p^{j-1}}} \right)$$

$$+ (y^{-p-1} - 1)x_2^p + \sum_{0<k\leq j} \frac{x_2^p - x_2^{p^{j-1}}}{1 - x_2^{p^{j-1}}}$$

$$= \sum_{0<k<j} \left( \frac{x_2^k - x_2^p}{1 - x_2^k} + \frac{x_2^p - x_2^{p^{j-1}}}{1 - x_2^{p^{j-1}}} \right)$$

$$+ (y^{-p-1} - 1)x_2^p$$

$$= \sum_{0<k<j} \frac{(1 - x_2^p)(x_2^k - x_2^{p^{j-1}})}{(1 - x_2^k)(1 - x_2^{p^{j-1}})}$$

$$+ x_2^{p+1}(y^{p^{j-1}} - y^p).$$

We also observe that

$$g(S_j) = g(A(m + 1)/I_2) \cdot \frac{x_2^{p^{j+1}}(1 - y^{p^{j-1}})}{1 - y} \cdot \frac{1 - x_2^{p^{j-1}}}{1 - x_2}$$

$$= g_{m+1}(t) \cdot \frac{x_2^{p^{j+1}}(1 - y^{p^{j-1}})(1 - x_2^{p^{j-1}})}{1 - x_2}.$$
Summing these three Poincaré series, we obtain

\[
g(\mathcal{S}_{0,1} \cup \mathcal{S}_{0,2} \cup \mathcal{S}_j) = \frac{g(\mathcal{S}_{0,1}) + g(\mathcal{S}_{0,2}) + g(\mathcal{S}_j)}{g_{m+1}(t)}
\]

\[
= \sum_{0<k<j} \frac{(1-x^p)(x^p - x^{p^{k-1}})}{(1-x^p)(1-x_2^{p^{k-1}})} + \frac{x^{p^{j+1}}}{1-x_2^{p^{j+1}}} + x^{p^{j+1}}(y^{p^{j-1}} - y^p) + \frac{x^{p^{j+1}}(1-y^{p^{j-1}})(1-x_2^{p^{j-1}})}{1-x_2}
\]

\[
= \sum_{0<k<j} \frac{(1-x^p)(x^p - x_2^{k-1})}{(1-x^p)(1-x_2^{k-1})} + \frac{x^{p^{j+1}}}{1-x_2} + \frac{x^{p^{j+1}}(1-x_2^{p^{j-1}} + y^{p^{j-1}}x_2^{p^{j-1}} - y^p - x_2y^{p^{j-1}} + x_2y^p)}{1-x_2}
\]

\[
= \sum_{0<k<j} \frac{(1-x^p)(x^p - x_2^{p^{k-1}})}{(1-x^p)(1-x_2^{p^{k-1}})} + \frac{x^{p^{j+1}}}{1-x_2} + \frac{x^{p^{j+1}}(1-x_2^{p^{j-1}} - y^{p^{j-1}}(x_2 - x_2^{p^{j-1}}) - y^p(1-x_2))}{1-x_2}
\]

On the other hand, Theorem 2.4 gives

\[
g(B_{m+1}) = \sum_{0<k\leq j+1} \frac{x^p - x_2^{p^{k-1}}}{(1-x^p)(1-x_2^{p^{k-1}})}
\]

\[
= \sum_{0<k<j} \frac{x^p - x_2^{k-1}}{(1-x^p)(1-x_2^{p^{k-1}})} + \frac{x^{p^{j+1}}}{1-x_2^{p^{j+1}}} + \frac{x^{p^{j+1}}(1-x_2^{p^{j-1}} - y^{p^{j-1}}x_2^{p^{j-1}} + y^p)}{1-x_2^{p^{j-1}}}
\]

below dimension \(|x^{p^{j+1}}x_2^{p^{j-1}}|\), so

\[
g(B_{m+1})(1-x^p) = \sum_{0<k<j} \frac{(x^p - x_2^{p^{k-1}})(1-x^p)}{(1-x^p)(1-x_2^{p^{k-1}})} + \frac{x^{p^{j+1}}}{1-x_2^{p^{j+1}}} + \frac{x^{p^{j+1}}(1-y^p)(1-x_2^{p^{j-1}})}{1-x_2^{p^{j+1}}}
\]
This means

\[
g(S_{0,1} \cup S_{0,2} \cup S_1) - g(B_{m+1})(1 - x^{p^j})
\]

\[
g_{m+1}(t)
\]

\[
x^{p^{j+1}}(1 - x^{p^{j+1}} - y^{p^{j+1}}(x_2 - x_2^{p^{j+1}}) - y^{p^j}(1 - x_2))
\]

\[
= \frac{x^{p^{j+1}}(1 - y^{p^j})(1 - x^{p^j})}{1 - x_2}
\]

\[
x^{p^{j+1}}(1 - y^{p^{j+1}} - x_2^{p^{j+1}}) - \frac{x^{p^{j+1}}(1 - y^{p^j} - x_2 + x_2 y^{p^j})}{1 - x_2}
\]

\[
\text{below dimension } |\hat{v}_1^{p^j(p+1)}|
\]

\[
= \frac{x^{p^{j+1}}(1 - x_2^{p^{j+1}})}{1 - x_2}.
\]

By Lemma 3.5, this means that \( B_{m+1} \) is \( j \)-free in the range claimed and that the first nontrivial \( \text{Ext}^1 \) has order \( p \).

(iii) To show that the generator of is \( \text{Ext}^1 \) the element specified, we first show that the indicated Massey product is defined.

For \( j > 1 \) and \( 1 < k < p \) we claim

\[
d(\hat{\beta}_{1+k^{p^{j-1}}/k^{p^{j-1}}}) = \langle \hat{\beta}_{1+p^{j-1}/p^{j-1}}, \hat{h}_{1,j}, \ldots, \hat{h}_{1,j} \rangle_{k-1}.
\]

This can be shown by induction on \( k \) and direct calculation as follows. Let

\[
s = \hat{v}_1^{p^{j-1}} - v_1^{p^{j-1}} \in T^{(j)}_m \subset G(m + 1).
\]

It follows that \( w = \hat{v}_2 - v_1 s \) is invariant. Note that its \( p^{j-1} \)th power does not lie in \( T^{(j)}_m \). Then we have

\[
\eta_R \left( \hat{\beta}_{1+k^{p^{j-1}}/k^{p^{j-1}}} \right) = \eta_R \left( \frac{v_2^{k^{p^{j-1}}}}{p v_1^{k^{p^{j-1}}}} \right)
\]

\[
= \sum_{0 < \ell \leq k} \binom{k^{p^{j-1}}}{\ell^{p^{j-1}}} \frac{v_2^{\ell^{p^{j-1}}}}{p v_1^{\ell^{p^{j-1}}}} \otimes s^{(k-\ell)p^{j-1}}
\]

\[
= \sum_{0 < \ell \leq k} \binom{k}{\ell} \frac{v_2^{\ell^{p^{j-1}}}}{p v_1^{\ell^{p^{j-1}}}} \otimes s^{(k-\ell)p^{j-1}}
\]

\[
= \sum_{0 < \ell \leq k} \binom{k}{\ell} \hat{\beta}_{1+\ell^{p^{j-1}}/\ell^{p^{j-1}}} \otimes s^{(k-\ell)p^{j-1}}
\]

\[
= \langle \hat{\beta}_{1+p^{j-1}/p^{j-1}}, \hat{h}_{1,j}, \ldots, \hat{h}_{1,j} \rangle_{k-1}.
\]

This means that our \( p \)-fold Massey product is defined.
We claim the first element in Ext$^1$ is represented by
\[
\sum_{0 \leq \ell < p} \frac{1}{p} \binom{p}{\ell} \beta_{1+\ell p^{j-1}/p^{j-1}} \otimes s(p-\ell) p^{j-1}
\]
\[
= \sum_{0 \leq \ell < p} \frac{1}{p} \binom{p}{\ell} \beta_{1+\ell p^{j-1}/p^{j-1}} \otimes \left( \beta_{p^{j-1}} - \beta_{p^{j-1}(p^{j-1})^2} \right)^{p-\ell}
\]
\[
= \sum_{0 \leq \ell < p} \frac{1}{p} \binom{p}{\ell} \beta_{1+\ell p^{j-1}/p^{j-1}} \otimes \beta_{p^{j-1}(p-\ell)}
\]
\[
= \beta_{1+1/p^{j-1}} \otimes \beta_{p^{j-1}} + \cdots
\]
The only element in $B_{m+1}$ in this dimension is $\beta_{1+p^{j-1}}$, which is primitive, so this element in Ext$^1$ is not trivial. \hfill \square

7. Higher Ext groups for $j = 1$

In this section we exhibit some calculations of Ext$^s(T^{(j)}_m \otimes B_{m+1})$ for $s > 0$. Recall the small descent spectral sequence, constructed in [Rav02, Theorem 1.17], which converges to Ext$^s(T^{(j)}_m \otimes B_{m+1})$ with
\[
E_{1}^{s,t} = E(\tilde{h}_j) \otimes P(\tilde{h}_j) \otimes \text{Ext}^{(s+1)}(T^{(j+1)}_m \otimes B_{m+1})
\]
with $\tilde{h}_j \in E_{1,0}^{1,0}$ and $\tilde{b}_j \in E_{2,0}^{2,0}$, and $d_r : E_{r,t}^s \to E_{r+t,r-1}^{s+1}$. In particular, $d_1$ is induced by the action of $\tilde{r}_p$ on $B_{m+1}$ for $s$ even and $\tilde{r}_{qp}$ for $s$ odd. The case $m = 0$ has already been treated in [Rav04, Chapter 7], so we may assume that $m > 0$. We examine the simplest case, $j = 1$. Recall that $B_{m+1}$ is 2-free below dimension $[\ell p^{j+1}/p]_2$ and Ext$^0(T^{(2)}_m \otimes B_{m+1})$ is the $A(m+1)$-module generated by
\[
\{ \beta_{i/t}^{j+1} : 0 < t \leq \min(i,p) \} \cup \{ \beta_{p^{j+1}/t} : p < t \leq p^2 \}
\]
by Theorem 6.1. Then the spectral sequence collapses from $E_2$. We can compute $d_1$ on elements (7.1) using Proposition B.2: The action of $\tilde{r}_p$ on Ext$^0(T^{(2)}_m \otimes B_{m+1})$ is given by $\tilde{r}_p \left( \beta_{i/t}^{j+1} \right) = \beta_{i-1/t}^{j+1}$ and $\tilde{r}_p \left( \beta_{p/t}^{j+1} \right) = 0$, and the action of $\tilde{r}_{qp}$ is obtained by composing $\tilde{r}_p$ up to unit scalar. In order to understand the behavior of $d_1$, the following picture for $p = 3$ may be helpful.

\[
\begin{array}{ccccccc}
\beta_1 & \tilde{r}_3 & \beta_2 & \tilde{r}_3 & \beta_3 & \tilde{r}_3 & \beta_3 \\
\beta_2/2 & \tilde{r}_3 & \beta_3/2 & \tilde{r}_3 & \beta_4/2 & \tilde{r}_3 \\
\beta_3/3 & \tilde{r}_3 & \beta_3/3 & \tilde{r}_3 & \beta_4/3 & \tilde{r}_3 \\
\end{array}
\]
Here each arrow represents the action of the Quillen operation $\tilde{r}_3$ up to unit scalar. For a general prime $p$, the analogous picture would show a directed graph with $2p$ components, two of which have $p$ vertices, and in which the arrow shows the action of the Quillen operation $\tilde{r}_p$ up to unit scalar. Each component corresponds to an $A(m + 1)$-summand of the $E_2$-term, with the caveat that $p\beta_{p/e} = \hat{\beta}_{p/e}$ and $v_1\beta_{i/e} = \hat{\beta}_{i/e-1}$. Notice that the entire configuration is $\mathbb{Z}_2$-periodic. Corresponding to the diagonal containing $\hat{\beta}_1$ in (7.2), the subgroup of $E_1$ generated by

$$\{ \beta_1, \beta_{2/2}, \beta_{3/3} \} \otimes E(h_{1,1}) \otimes P(b_{1,1})$$

reduces on passage to $E_2$ to simply $\{ \hat{\beta}_1 \}$. Similarly, the subset

$$\{ \beta_2, \beta_{3/2} \} \otimes E(h_{1,1}) \otimes P(b_{1,1})$$

reduces to $\{ \beta_2, \beta_{3/2} \hat{h}_{1,1} \} \otimes P(b_{1,1})$, where

$$\beta_{3/2} \hat{h}_{1,1} = (h_{1,1}, \hat{h}_{1,1}, \hat{\beta}_2)$$

and $\hat{h}_{1,1}(\beta_{3/2} \hat{h}_{1,1}) = (\hat{h}_{1,1}, \hat{h}_{1,1}, \beta_{3/2}) = (\hat{h}_{1,1}, \hat{h}_{1,1}, \hat{h}_{1,1}) \beta_{3/2} = \hat{h}_{1,1} \beta_{3/2}$.

These observations give us the following result.

**Proposition 7.3. Structure of $\text{Ext}(\mathcal{T}_m^{(1)} \otimes B_{m+1})$.** In dimensions less than $|\mathbb{Z}_2^{p+1}/v_1^2|$, $\text{Ext}(\mathcal{T}_m^{(1)} \otimes B_{m+1})$ is a free module over $A(m + 1)/I_2$ with basis

$$\{ \beta_{1+pi}; \beta_{p+pi}; \beta_{p^2/k} \} \oplus \tilde{h}_{1,1} \left\{ \beta_{p^l/r}; \beta_{p+pi/r}; \beta_{p^2/p} \right\},$$

where $0 \leq i < p$, $1 \leq k \leq p^2 - p + 1$, $p^2 - p + 2 \leq l \leq p^2$, $2 \leq s \leq p$, $1 \leq t \leq p - 1$ and $p \leq u \leq 2p - 2$, subject to the caveat that $v_1\beta_{p/e} = \beta_{p/e-1}$ and $p\beta_{p/e} = \beta_{p/e}$.

In particular $\text{Ext}^0(\mathcal{T}_m^{(1)} \otimes B_{m+1})$ has basis

$$\{ \beta_{1+pi}, \cdots, \beta_{p+pi}; \beta_{p+pi/p}, \cdots, \beta_{p+pi+1}; \beta_{p^2/p}, \cdots, \beta_{p^2/p} \}.$$ 

Note that for $m > 0$, this range of dimensions exceeds $p|\bar{v}_3|$.

**Appendix A. Some results on binomial coefficients**

Fix a prime number $p$.

**Definition A.1. $\alpha(n)$, the sum of the $p$-adic digits of $n$.** For a nonnegative integer $n$, $\alpha(n)$ denotes sum of the digits in the $p$-adic expansion of $n$, i.e., for $n = \sum_{i \geq 0} a_i p^i$ with $0 \leq a_i \leq p - 1$, we define $\alpha(n) = \sum_{i \geq 0} a_i$.

As before, let $\nu_p(n)$ denote the $p$-adic valuation of $n$, i.e., the exponent that makes $n$ a $p$-local unit multiple of $p^{\nu_p(n)}$. When the integer is a binomial coefficient $\binom{j}{i}$, we will write $\nu_p\left(\binom{j}{i}\right)$ instead of $\nu_p\left(\binom{j}{i}\right)$. Then we have
Lemma A.2. $p$-adic valuation of a binomial coefficient.

$$q v_p \left( \binom{n}{k} \right) = \alpha(k) + \alpha(n-k) - \alpha(n)$$

where $q = p - 1$. In particular,

$$q v_p \left( \binom{n}{p^j} \right) = 1 + \alpha(n - p^j) - \alpha(n).$$

Proof. Recall that $0 \leq s$ is a prime and that a positive integer $s$ needs are the followings:

We use this lemma to determine the number how many times a binomial coefficient is divisible by a prime $p$. For example, we have

Lemma A.3. Divisibility of a binomial coefficient. Assume that $p \mid a$ and $0 < n \leq \ell$. Then the binomial coefficient $\binom{ap^\ell + n - 1}{n}$ is divisible by $p^{\ell+1-n}$.

Proof. Since $a \not\equiv 0 \pmod{p}$, we have $\alpha(a - 1) = \alpha(a) - 1$. Let $m = \nu_p(n)$ and $n = n'p^m$. Then $\alpha(n' - 1) = \alpha(n') - 1$, and we have

$$q v_p \left( \binom{ap^\ell + n - 1}{n} \right) = q v_p \left( \binom{ap^\ell + n'p^m - 1}{n'p^m} \right)$$

$$= \alpha(n'p^m) + \alpha(ap^\ell - 1) - \alpha(ap^\ell + n'p^m - 1)$$

$$= \alpha(n') + \alpha(a - 1) + q\ell - \alpha(ap^\ell + n'p^m - 1) - qm$$

$$= \alpha(n') + \alpha(a - 1) + q\ell - \alpha(a) - \alpha(n' - 1) - qm$$

$$= q(\ell - m) \geq q(\ell + 1 - n).$$

We consider this type of binomial coefficients in Proposition 4.4. The other types we need are the followings:

Lemma A.4. Divisibility of another binomial coefficient. Assume that $p$ is a prime and that a positive integer $s$ is expressed as $s = s_1p^\ell + s_0 > 0$ with $0 \leq s_0 < p^\ell$. Then we have $\nu_p \left( \binom{s}{p^\ell} \right) = \nu_p(s_1)$. In particular, we have $p^\alpha \mid \binom{s}{p^\ell}$ if $s \equiv 0, 1, \ldots, p^\ell - 1 \pmod{p^{n+\ell}}$.

Proof. Observe that

$$q v_p \left( \binom{s}{p^\ell} \right) = \alpha(p^\ell) + \alpha(s - p^\ell) - \alpha(s)$$

$$= 1 + \alpha((s_1 - 1)p^\ell + s_0) - \alpha(s_1p^\ell + s_0)$$

$$= \alpha(1) + \alpha(s_1 - 1) - \alpha(s_1)$$

$$= q v_p(s_1).$$
This implies that $\nu_p(s^*) = n \iff s \equiv s_0 \text{ mod } (p^{n+\ell})$.

In Appendix B it is required to know how many times the binomial coefficient $\binom{i-1}{p^{i-1}-1}$ is divisible by $p$.

For $0 < i < p^{j-1}$ it is clear that $\binom{i-1}{p^{i-1}-1} = 0$. For $i \geq p^{j-1}$, the number $\nu_p\left(\binom{i-1}{p^{i-1}-1}\right)$ can be determined explicitly in the following results.

**Proposition A.5. A third divisibility statement.** For $i \geq p^{j-1}$, define non-negative integers $i_0$ and $i_1$ by

\begin{equation}
(A.6) \quad i = i_1 p^{j-1} + i_0 \quad (i_1 > 0 \text{ and } 0 \leq i_0 < p^{j-1}).
\end{equation}

Then we have

1. $\binom{i-1}{p^{i-1}-1}$ is divisible by $p \iff i_0 \neq 0$;
2. More generally, $\binom{i-1}{p^{i-1}-1}$ is divisible by $p^{j-k}$ ($0 \leq k < j$) \iff

\begin{equation}
(A.7) \quad \nu_p(i_0) \leq k - 1 + \nu_p(i_1).
\end{equation}

or equivalently $i_0 \neq 0$ and $p^{k+\nu_p(i_1)} \mid i_0$. 

In particular, the inequality (A.7) is automatically satisfied if $\nu_p(i_1) \geq j - k - 1$.

**Proof.** Observe that

\[
\nu_p\left(\binom{i-1}{p^{i-1}-1}\right) = \nu_p\left(p^{j-1}\right) + \nu_p\left(\binom{i}{p^{j-1}-1}\right) - \nu_p(i)
\]

\[
= (j - 1) + \nu_p(i_1) - \left\{ \begin{array}{ll}
(j - 1 + \nu_p(i_1)) & \text{if } i_0 = 0 \\
\nu_p(i_0) & \text{if } i_0 \neq 0
\end{array} \right. 
\]

\[
= \left\{ \begin{array}{ll}
0 & \text{if } i_0 = 0 \\
(j - 1 + \nu_p(i_1) - \nu_p(i_0) & \text{if } i_0 \neq 0
\end{array} \right.
\]

If $i_0 \neq 0$, then we have $j - 1 + \nu_p(i_1) - \nu_p(i_0) > 0$ since $\nu_p(i_0) \leq j - 2$, and so the binomial coefficient is divisible by $p$. Since $i_0 = 0$ is equivalent to $p^{j-1} \mid i$, the statement (1) follows.

The condition $p^{j-k} \mid \binom{i-1}{p^{i-1}-1}$ is equivalent to the inequality $\nu_p\left(\binom{i-1}{p^{i-1}-1}\right) \geq j - k$, and if we suppose that $j - k > 0$ then this inequality gives (A.7).

Note that (A.7) is always satisfied if $\nu_p(i_1) \geq j - k - 1$ since $\nu_p(i_0) \leq j - 2$ by definition.

The following is the obvious translation of Proposition A.5.

**Corollary A.8. A fourth divisibility statement.** Let $i_0$ and $i_1$ be as in (A.6) and assume that $p^{j-1} < i \leq p^{j-1+\ell}$. Then, we have $p^{j-k} \mid \binom{i-1}{p^{i-1}-1}$ for $0 \leq k < j$ \iff

\[
\nu_p(i_0) \leq k - 1 + \nu_p(i_1) \quad \text{with } 0 \leq \nu_p(i_1) \leq m.
\]

**Proof.** The given range $p^{j-1} < i \leq p^{j-1+\ell}$ means that $0 \leq \nu_p(i_1) \leq m$ and the result follows from Proposition A.5.
APPENDIX B. QUILLEN OPERATIONS ON $\beta$-ELEMENTS

In this section we discuss the action of the Quillen operations $\widehat{r}^j$ for $j > 0$ on the $\beta$-elements.

First we consider the following easy cases.

**Proposition B.1. Primitive $\beta$-elements.** For $i > 0$, the elements $\widehat{\beta}_{i/t}$ are primitive if $0 < t \leq p^{\nu(i)}$, i.e., it satisfies $\widehat{r}_t(\widehat{\beta}_{i/t}) = 0$ for all $\ell \geq 0$.

**Proof.** Set $\nu_p(i) = n$ and $i = i' p^n$. By direct calculations we have

$$\eta_R \left( \frac{c_2^i}{pv_1^i} \right) = \frac{(c_2^i + v_1^{p^n} c_1^{n+1} - v_1^{p^n+1} c_1^{n+1})^i}{pv_1^i} = \frac{c_2^i}{pv_1^i}.$$  

$\square$

For the other cases, the Quillen operation $\widehat{r}^j$ is computed as follows:

**Proposition B.2. Quillen operations on $\beta$-elements.** When $j > 0$, we have

$$\widehat{r}^j(\widehat{\beta}_{i/t}) = \binom{i-1}{p^j-1} \widehat{\beta}_{i-p^j-1/t-p^j-1}$$  \text{ for } t < p^j-1 + p^{n+2}.

**Proof.** First assume that $m > 0$. Observe that

$$\eta_R(\widehat{\beta}_{i/t}) = \eta_R \left( \frac{c_2^i}{ipv_1^i} \right) = \sum_{0 \leq k \leq \ell \leq i} \binom{i}{\ell} \binom{\ell}{k} \frac{c_2^i \ell-k (v_1^{p^m} c_1^k)}{ipv_1^i}.$$  

Since $\widehat{r}^j(\widehat{\beta}_{i/t})$ is the coefficient of $\widehat{c}_2^i$ in the above, we need to consider the terms satisfying $p(\ell - k) + k = p^j$. Note that $k$ must be divisible by $p$ and that we may set $k = pn$. Thus we have

$$p^j = p(\ell - pm) + pm.$$  

Now let

$$\ell(n) = \ell = p^j-1 + qn \quad \text{where } q = p - 1$$  \text{ and }  

$$g(n) = t - \ell + k - pm.$$  

Then we have

$$\widehat{r}^j(\widehat{\beta}_{i/t}) = \sum_{0 \leq n \leq p^j-1} (-1)^{pn} \binom{i-1}{\ell(n)} \binom{\ell(n)}{np} \frac{c_2^i \ell(n)}{(i - \ell(n))pv_1^{g(n)}}.$$
Given our assumption about \( t \), the only value of \( n \) satisfying \( g(n) > 0 \) is \( n = 0 \), which gives

\[
\hat{\tau}_p^i(\beta_{i/t}) = \binom{i-1}{p^{i-1}} \frac{v_p^{i-p^i-1}}{(i-p^{i-1})p^{i-p^i-1}v_1^lp^{i-p^i-1}}.
\]

The proof for \( m = 0 \) is more complicated. Observe that

\[
\psi(\beta_{i/t}) = \sum_{0 \leq k \leq \ell} \sum_{r \geq 0} (-1)^{k+r} \binom{i-1}{\ell} \binom{\ell(n,r)-1}{r} \binom{t+r-1}{r} p^{\ell\rho'_{i-t}(n,r)} (i \ell v_1^{(\ell-k)+k+r} p^{i+p-\ell+k-pk}),
\]

which shows that \( \hat{\tau}_p^i(\beta_{i/t}) \) is equal to

\[
\sum_{0 \leq n \leq p^{i-1}} \sum_{0 \leq r \leq np^i} (-1)^p \binom{i-1}{\ell(n,r)-1} \binom{\ell(n,r)-1}{r} \frac{p^{\ell\rho'_{i-t}(n,r)}}{(np-r)v_1^g(n,r)},
\]

where \( \ell(n,r) = p^{i-1} + np - r \) and \( g(n,r) = t - p^{i-1} - np^i + r(p+1) \). If \( p^r | (np-r) \) for a positive \( r \), then we may put \( r = sp \) and \( n \geq p^{sp-1} + s \) for a positive \( s \) and the exponent of \( v_1 \) is not positive since

\[
g(n,r) \leq t - p^{i-1} - (p^{sp-1} + s)(p^2 - 1) + sp(p+1)
\]

\[
= t - p^{i-1} - (p+1)(p^{sp} - p^{p^i-1} - s)
\]

\[
\leq t - p^{i-1} - (p+1)(p^p - p^{p^i-1} - 1)
\]

\[
\leq t - p^{i-1} - (p^2 - 1).
\]

Thus, the nontrivial term arises only when \( r = 0 \). We can see that it is also required that \( n = 0 \) by the same reason as the \( m > 0 \) case, and the result follows.

To know the condition of triviality of \( \hat{\tau}_p^i \) in Proposition B.2, we need the results on the \( p \)-adic valuation of binomial coefficients obtained in Appendix A. In particular, we have

**Proposition B.3. Some trivial actions of Quillen operations.** Assume that \( p^{i-1} < i \leq p^{i+1} \) and \( t < p^{i-1} + p^{m+2} \). Then we have the following trivial Quillen operations:

1. \( \hat{\tau}_p^i(\beta_{i/t}) (\ell \geq j) \) for \( 0 < t \leq \min(i, p^{i-1}) \);
2. \( \hat{\tau}_p^i(\beta_{ap^i+b/t}) (\ell \geq j) \) for \( p^{i-1} < t \leq p^i \) and \( 0 \leq b < p^{i-1} \).

**Proof.** We will show the following Quillen operations on \( p^k \beta_{i/t} \) are trivial:

- (a) \( \hat{\tau}_p^i (\ell \geq j) \) for \( 0 < t \leq \min(i, p^{i-1}) \) and \( k \geq 0 \);
- (b) \( \hat{\tau}_p^i (\ell \geq j) \) for \( p^{i-1} < t \leq p^i \), \( i = ap^i + bp^k \) with \( p \nmid a, p \nmid b \) and \( 0 \leq k < j-1 \);
- (c) \( \hat{\tau}_p^i (\ell \geq 0) \) for \( p^{i-1} < t \leq p^i \), \( i = ap^i \) with \( 0 < a \leq p \) and \( j = k \).

For the case (1), note that

\[
\hat{\tau}_p^i(p^k \beta_{i/t}) = \binom{i-1}{p^{i-1} - 1} \frac{v_p^{i-p^i-1}}{p^{i-k}v_1^{i-p^i-1}}.
\]

by Proposition B.2, which is clearly trivial when \( 0 < t \leq p^{i-1}(\leq p^{i-1}) \). Even if \( p^{i-1} < t \leq i \), it is trivial when the binomial coefficient \( \binom{i-1}{p^{i-1}-1} \) is divisible by \( p^{i-k} \), or equivalently when the inequality (A.7) holds.
When $0 < k < j$, by the assumption we have
\[ p^{j-1} < i_1 p^{j-1} + i_0 \leq p^{j+1} \]
(where $\nu_p(i_0) < j-1$ by definition) and $\nu_p(i_1) \leq 2$. Note that if $k > 0$ and $p^k \not| \ i$ then $p^k \beta_{ij/t}$ itself is trivial and that we may assume that $\nu_p(i) \geq k$. These observations suggest that the only case satisfying the inequality (A.7) is $(\nu_p(i_1), \nu_p(i_0)) = (1, k)$, which gives the case (b).

When $j = k$, the Quillen operation $\hat{r}_p(p^j \beta_{ij/t})$ is clearly trivial and $p^j \beta_{ij/t}$ is nontrivial only if $p^j \mid i$, which gives the case (c).

For the case (b) and (c), observe that the Quillen operation $\hat{r}_{p^{j+1}}(p^k \beta_{ij/t})$ is a unit scalar multiple of $\beta_{i-p^j t-p^j}$ and $p^k \beta_{ij/t}$ is not in $L_t(B_{m+1})$, which means that the condition $t \leq p^j$ is required. Combining (b) and (c) gives the case (2).

Note that no linear combination of $\beta$-elements can be killed by $\hat{r}_p$ since the $\hat{r}_p$-image has different exponents of $\hat{v}_2$ or $v_1$ if $\beta_{i_1/t_1} \neq \beta_{i_2/t_2}$.

\[ \square \]

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