A solution to the Arf-Kervaire invariant problem

Second Abel Conference: A Mathematical Celebration of John Milnor

February 1, 2012

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University of Virginia
Mike Hopkins
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Classifying exotic spheres
Pontryagin's early work
Exotic spheres as framed manifolds

The Arf-Kervaire invariant

The main theorem

Our strategy
Ingredients of the proof
The spectrum \(\Omega\)
How we construct \(\Omega\)
The slice spectral sequence

Mike Hill, myself and Mike Hopkins
Photo taken by Bill Browder
February 11, 2010
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• Kervaire and Milnor’s *Groups of homotopy spheres, I*, 1963.

For example, for $n = 1, 2, 3, \cdots, 18$, it will be shown that the order of the group $\Theta_n$ is respectively:

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(i) Their answer was given in terms of the stable homotopy groups of spheres, which remain a mystery to this day.

(ii) There was an ambiguous factor of two in dimensions congruent to 1 mod 4. The solution to that problem is the subject of this talk.
Pontryagin’s early work on homotopy groups of spheres

Back to the 1930s

Pontryagin’s approach to continuous maps $f: S^n+k \to S^k$ was
• Assume $f$ is smooth. We know that any map $f$ can be continuously deformed to a smooth one.
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Pontryagin’s early work (continued)

Let $D^k$ be the closure of an open ball around a regular value $y \in S^k$. 

\[ M^n \times D^k \overset{f}{\longrightarrow} S^{n+k} \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ V^{n+k} \overset{f}{\longrightarrow} D^k \]

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Let $D^k$ be the closure of an open ball around a regular value $y \in S^k$. If it is sufficiently small, then $V^{n+k} = f^{-1}(D^k) \subset S^{n+k}$ is an $(n+k)$-manifold homeomorphic to $M \times D^k$. 

\[
\begin{array}{c}
S^{n+k} \xrightarrow{f} S^k \\
\uparrow \quad \uparrow \\
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A local coordinate system around around the point $y \in S^k$ pulls back to one around $M$ called a framing.
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\[ M^n \times D^k \leftarrow V^{n+k} \rightarrow D^k \]

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\[ f \]

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There is a way to reverse this procedure. A framed manifold \( M^n \subset S^{n+k} \) determines a map \( f : S^{n+k} \rightarrow S^k \).
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Two maps \( f_1, f_2 : S^{n+k} \to S^k \) are **homotopic** if there is a continuous map \( h : S^{n+k} \times [0, 1] \to S^k \) (called a **homotopy** between \( f_1 \) and \( f_2 \)) such that
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Pontryagin’s early work (continued)
Here is an example of a framed cobordism for $n = k = 1$. 

Framed cobordism
Pontryagin’s early work (continued)

Let $\Omega_{n,k}^{fr}$ denote the cobordism group of framed $n$-manifolds in $\mathbb{R}^{n+k}$, or equivalently in $S^{n+k}$. 

Pontryagin's construction leads to a homomorphism $\Omega_{n,k}^{fr} \to \pi_{n+k}(S^k)$. Pontryagin's Theorem (1936) states that the above homomorphism is an isomorphism in all cases. Both groups are known to be independent of $k$ for $k > n$. We denote the resulting stable groups by simply $\Omega^{fr}_n$ and $\pi_{S^n}$.

The determination of the stable homotopy groups $\pi_{S^n}$ is an ongoing problem in algebraic topology. Experience has shown that unfortunately its connection with framed cobordism is not very helpful. It is not used in the proof of our theorem.
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*The above homomorphism is an isomorphism in all cases.*

Both groups are known to be independent of $k$ for $k > n$. We denote the resulting stable groups by simply $\Omega_n^{fr}$ and $\pi_n^S$.

The determination of the stable homotopy groups $\pi_n^S$ is an ongoing problem in algebraic topology. Experience has shown that unfortunately its connection with framed cobordism is not very helpful. It is not used in the proof of our theorem.
Exotic spheres as framed manifolds

Into the 60s again
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Exotic spheres as framed manifolds (continued)

Two framings of an exotic sphere $\Sigma^n \subset S^{n+k}$ differ by a map to $\text{SO}(k)$, and this map does not depend on the differentiable structure on $\Sigma^n$. Varying the framing on the standard sphere $S^n$ leads to a homomorphism $\pi_n \to \pi_{n+k}$ called the Hopf-Whitehead $J$-homomorphism. It is well understood by homotopy theorists.
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Heinz Hopf 1894-1971

George Whitehead 1918-2004
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Thus we get a homomorphism

\[ \Theta_n \xrightarrow{p} \pi_n S / \text{Im } J. \]
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\[ \Theta_n \xrightarrow{p} \pi_n^S / \text{Im } J. \]

The bulk of the Kervaire-Milnor paper is devoted to studying its kernel and cokernel using surgery. The two questions are closely related.

• The map \( p \) is onto iff every framed \( n \)-manifold is cobordant to a sphere, possibly an exotic one.
• It is one-to-one iff every exotic \( n \)-sphere that bounds a framed manifold also bounds an \((n+1)\)-dimensional disk and is therefore diffeomorphic to the standard \( S_n \).

They denote the kernel of \( p \) by \( bP_n+1 \), the group of exotic \( n \)-spheres bounding parallelizable \((n+1)\)-manifolds.

Behrens called this group \( \Theta bP_n \).
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Hence we have an exact sequence

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- *The homomorphism* \( p \) *above is onto except possibly when* \( n = 4m + 2 \) *for* \( m \in \mathbb{Z} \), *and then the cokernel has order at most 2.*
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- \( bP_{4m} \) is a certain cyclic group. Its order is related to the numerator of the \( m \)th Bernoulli number.
- The order of \( bP_{4m+2} \) is at most 2.
- \( bP_{4m+2} \) is trivial iff the cokernel of \( p \) in dimension \( 4m + 2 \) is nontrivial.

We now know the value of \( bP_{4m+2} \) in every case except \( m = 31 \).
In other words have a 4-term exact sequence

\[ 0 \rightarrow \Theta_{4m+2} \xrightarrow{p} \pi^{S}_{4m+2}/\text{Im } J \rightarrow \mathbb{Z}/2 \rightarrow bP_{4m+2} \rightarrow 0 \]
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To say more about this we need to define the Kervaire invariant of a framed manifold.
A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
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The slice spectral sequence

The Arf invariant of a quadratic form in characteristic 2

Back to the 1940s

Let $\lambda$ be a nonsingular anti-symmetric bilinear form on a free abelian group $H$ of rank $2^n$ with mod 2 reduction $H$. It is known that $H$ has a basis of the form $\{a_i, b_j : 1 \leq i, j \leq n\}$ with $\lambda(a_i, a_i') = 0$, $\lambda(b_j, b_j') = 0$, and $\lambda(a_i, b_j) = \delta_{i, j}$. 
The Arf invariant of a quadratic form in characteristic 2

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Cahit Arf 1910-1997

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$$\lambda(a_i, a_{i'}) = 0 \quad \lambda(b_j, b_{j'}) = 0 \quad \text{and} \quad \lambda(a_i, b_j) = \delta_{i,j}.$$
In other words, $\overline{H}$ has a basis for which the bilinear form’s matrix has the symplectic form

$$
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 1 \\
1 & 0 \\
\ddots & \ddots \\
0 & 1 \\
1 & 0
\end{bmatrix}.
$$
The Arf invariant of a quadratic form in characteristic 2 (continued)

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In 1941 Arf proved that this invariant (along with the number $n$) determines the isomorphism type of $q$. 

The Arf invariant of a quadratic form in characteristic 2 (continued)
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Alan M. Turing, codebreaker and an Enigma Code Machine

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Money talks: Arf’s definition republished in 2009

Cahit Arf 1910-1997
The Kervaire invariant of a framed \((4m + 2)\)-manifold

Into the 60s a third time
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Into the 60s a third time
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The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

For $M = T^2 \subset S^3$ and $x \in H_1(T^2; \mathbb{Z}/2)$, $q(x)$ is the number of full twists in a cylinder $V$ neighboring a curve representing $x$. 


The Kervaire invariant of a framed $(4m + 2)$-manifold
(continued)

For $M = T^2 \subset S^3$ and $x \in H_1(T^2; \mathbb{Z}/2)$, $q(x)$ is the number of full twists in a cylinder $V$ neighboring a curve representing $x$. This function is not additive!
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Again, let $$M$$ be a $$2m$$-connected smooth closed framed manifold of dimension $$4m + 2$$, and let $$H = H_{2m+1}(M;\mathbb{Z})$$. Each $$x \in H$$ is represented by an embedding $$S^{2m+1} \hookrightarrow M$$. 

1.25

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Kervaire defined a quadratic refinement \(q\) on its mod 2 reduction \(H\) in terms of each sphere's normal bundle. The Kervaire invariant \(\Phi(M)\) is defined to be the Arf invariant of \(q\).

Recall the Kervaire-Milnor 4-term exact sequence

\[0 \to \Theta_{4m+2} \to \pi_{S^{4m+2}}/\text{Im} J \to \mathbb{Z}/2 \to bP_{4m+2} \to 0\]

Kervaire-Milnor Theorem (1963)

\[bP_{4m+2} = 0\] if there is a smooth framed \((4m + 2)\)-manifold \(M\) with \(\Phi(M)\) nontrivial.
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Again, let $M$ be a $2m$-connected smooth closed framed manifold of dimension $4m + 2$, and let $H = H_{2m+1}(M; \mathbb{Z})$. Each $x \in H$ is represented by an embedding $S^{2m+1} \hookrightarrow M$. $H$ has an antisymmetric bilinear form $\lambda$ defined in terms of intersection numbers.

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Kervaire defined a quadratic refinement \(q\) on its mod 2 reduction \(\overline{H}\) in terms of each sphere’s normal bundle. The Kervaire invariant \(\Phi(M)\) is defined to be the Arf invariant of \(q\).

Recall the Kervaire-Milnor 4-term exact sequence

\[
0 \longrightarrow \Theta_{4m+2} \longrightarrow p \longrightarrow \pi_{4m+2}^S / \text{Im} J \longrightarrow \mathbb{Z}/2 \longrightarrow bP_{4m+2} \longrightarrow 0
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The Kervaire invariant of a framed \((4m + 2)\)-manifold (continued)

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**Kervaire-Milnor Theorem (1963)**

\(bP_{4m+2} = 0\) iff there is a smooth framed \((4m + 2)\)-manifold \(M\) with \(\Phi(M)\) nontrivial.
The Kervaire invariant of a framed $(4m + 2)$-manifold (continued)

What can we say about $\Phi(M)$?
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Pontryagin (1930’s)

[Diagram of a framed torus]

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The Kervaire invariant of a framed \((4m + 2)\)-manifold (continued)

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\[ X = N/\partial N \]
\(\text{(a triangulable manifold)}\)
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This construction generalizes to higher $m$, but Kervaire’s proof that the boundary is exotic does not.
The Kervaire invariant of a framed \((4m + 2)\)-manifold (continued)

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1930-2000

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Bill Browder
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- $\theta_j$ is known to exist for $1 \leq j \leq 5$, i.e., in dimensions 2, 6, 14, 30 and 62.
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- \(\theta_j\) is known to exist for \(1 \leq j \leq 5\), i.e., in dimensions 2, 6, 14, 30 and 62. In other words, \(bP_2, bP_6, bP_{14}, bP_{30}\) and \(bP_{62}\) are all trivial.
And then ...
And then ... the problem went viral!

A wildly popular dance craze

Can you do the Arf Invariant?
Is it a jig or a reel?

Drawing by Carolyn Snaith 1981
London, Ontario
Speculations about $\theta_j$ after Browder’s theorem

In the decade following Browder’s theorem, many topologists tried without success to construct framed manifolds with nontrivial Kervaire invariant in all such dimensions, i.e., to show that $bP_{2j+1-2} = 0$ for all $j > 0$. 
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Mark Mahowald
A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

Background and history

Classifying exotic spheres
Pontryagin's early work
Exotic spheres as framed manifolds

The Arf-Kervaire invariant

The main theorem

Our strategy

Ingredients of the proof
The spectrum \(\Omega\)
How we construct \(\Omega\)
The slice spectral sequence

Mark Mahowald’s sailboat
Mark Mahowald’s sailboat
After Browder’ theorem (continued)

Vic Snaith and Bill Browder in 1981
Photo by Clarence Wilkerson
After Browder’ theorem (continued)

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After 1980, the problem faded into the background because it was thought to be too hard.
After Browder’ theorem (continued)

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Fast forward to 2009
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“As ideas for progress on a particular mathematics problem atrophy it can disappear. Accordingly I wrote this book to stem the tide of oblivion.”
Snaith’s book (continued)

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“For a brief period overnight we were convinced that we had the method to make all the sought after framed manifolds - a feeling which must have been shared by many topologists working on this problem. All in all, the temporary high of believing that one had the construction was sufficient to maintain in me at least an enthusiastic spectator’s interest in the problem.”
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- **Stable homotopy theoretic formulation**: It says that certain long sought hypothetical maps between high dimensional spheres do not exist.

There were several unsuccessful attempts in the 1970s to prove the opposite of what we have proved, namely that $bP_{4m+2} = 0$ for all $j > 0$. 
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**Main Theorem**

The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1} - 2 + n}(S^n)$ for large $n$ do not exist for $j \geq 7$. 

The $\theta_j$ in the theorem is the name given to a hypothetical map between spheres represented by a framed manifold with nontrivial Kervaire invariant. It follows from Browder's theorem of 1969 that such things can exist only in dimensions that are $2$ less than a power of $2$.

**Corollary**

The Kervaire-Milnor group $bP_{2^{j+1} - 2 + n}$ is nontrivial for $j \geq 7$. It is known to be trivial for $1 \leq j \leq 5$. The case $j = 6$, i.e., $bP_{126}$, is still open.
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Questions raised by our theorem

Adams spectral sequence formulation. We now know that the $h^j$ for $j \geq 7$ are not permanent cycles, so they have to support nontrivial differentials. We have no idea what their targets are.

Unstable homotopy theoretic formulation. In 1967 Mahowald published an elaborate conjecture about the role of the $\theta_j$ (assuming that they all exist) in the unstable homotopy groups of spheres. Since they do not exist, a substitute for his conjecture is needed. We have no idea what it should be.

Our method of proof offers a new tool, the slice spectral sequence, for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future. I will illustrate it at the end of the talk.
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- We use methods of stable homotopy theory, which means we use spectra instead of topological spaces. Roughly speaking, spectra are to spaces as integers are to natural numbers. Instead of making addition formally invertible, we do the same for suspension. While a space $X$ has a homotopy group $\pi_n(X)$ for each positive integer $n$, 

\[ \pi_0(S) \text{ is an element of this group for } n = 2j + 1 - 2. \]
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- We use methods of **stable homotopy theory**, which means we use spectra instead of topological spaces. Roughly speaking, spectra are to spaces as integers are to natural numbers. Instead of making addition formally invertible, we do the same for suspension. While a space $X$ has a homotopy group $\pi_n(X)$ for each positive integer $n$, a spectrum $X$ has an abelian homotopy group $\pi_n(X)$ **defined for every integer $n$**.
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Our proof has several ingredients.

- We use methods of **stable homotopy theory**, which means we use spectra instead of topological spaces. Roughly speaking, spectra are to spaces as integers are to natural numbers. Instead of making addition formally invertible, we do the same for suspension. While a space $X$ has a homotopy group $\pi_n(X)$ for each positive integer $n$, a spectrum $X$ has an abelian homotopy group $\pi_n(X)$ defined for every integer $n$.

For the sphere spectrum $S^0$, $\pi_n(S^0)$ (previously denoted by $\pi_n^S$) is the usual homotopy group $\pi_{n+k}(S^k)$ for $k > n + 1$. 
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Ingredients of the proof (continued)

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John Milnor

Sergei Novikov

Dan Quillen 1940–2011
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The spectrum $\Omega$

We will produce a map $S^0 \to \Omega$, where $\Omega$ is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.
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(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of $k$ modulo 256.

(iii) Gap Theorem. $\pi_k(\Omega) = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence, which I will illustrate at the end of the talk.
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A solution to the Arf-Kervaire invariant problem

Mike Hill
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How we construct $\Omega$

Our spectrum $\Omega$ will be the fixed point spectrum for the action of $C_8$ (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$. To construct it we start with the complex cobordism spectrum $MU$. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of $C_2$ defined by complex conjugation. The fixed point set of this action is the set of real points, known to topologists as $MO$, the unoriented cobordism spectrum. In this notation, $U$ and $O$ stand for the unitary and orthogonal groups.
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To get a $C_8$-spectrum, we use the following general construction for getting from a space or spectrum $X$ acted on by a group $H$ to one acted on by a larger group $G$ containing $H$ as a subgroup.
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The corresponding slice spectral sequence