Browder’s work on Arf-Kervaire invariant problem

Panorama of Topology
A Conference in Honor of William Browder

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Browder’s theorem and its impact

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The Kervaire invariant of framed manifolds and its generalization*

By William Browder

In 1960, Kervaire [11] introduced an invariant for almost framed \((4k + 2)\)-manifolds, \((k \neq 0, 1, 3)\), and proved that it was zero for framed 10-manifolds,
In 1969 Browder proved a remarkable theorem about the Kervaire invariant.

**Browder’s Theorem (1969)**

*The Kervaire invariant of a smooth framed \((4m + 2)\)-manifold \(M\) can be nontrivial only if \(m = 2^{j-1} - 1\) for some \(j > 0\).*
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The Kervaire invariant of a smooth framed \((4m + 2)\)-manifold \(M\) can be nontrivial only if \(m = 2^{j-1} - 1\) for some \(j > 0\). This happens iff the element \(h_j^2\) is a permanent cycle in the Adams spectral sequence.

Browder’s theorem on the Arf-Kervaire invariant problem

Mike Hill
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Some early homotopy theory
Classifying exotic spheres
Exotic spheres as framed manifolds

The Arf-Kervaire invariant

The Arf invariant
The Kervaire invariant
Some theorems about \(\phi(M)\)

Browder’s theorem
The quadratic operation
Wu classes
The Browder spectrum
The homotopy type of \(BR_{2m+2}\)
Browder’s theorem and its impact (continued)

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\[
q : H_{2m+1}(M; \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}
\]

\[
h^2_j \in \text{Ext}_{\mathcal{A}}^{2, 2j+1}(\mathbb{Z}/2, \mathbb{Z}/2)
\]

\[
\theta_j \in \pi_{2j+1, -2}S^0
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This result established a link between surgery theory, specifically an unanswered question in the Kervaire-Milnor classification of exotic spheres, and stable homotopy theory, specifically the Adams spectral sequence.

This connection made the problem of constructing a smooth framed manifold with nontrivial Kervaire invariant in dimension $2^{j+1} - 2$ a cause celebre in algebraic topology throughout the 1970s.
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This connection made the problem of constructing a smooth framed manifold with nontrivial Kervaire invariant in dimension \(2^{j+1} - 2\) a cause celebre in algebraic topology throughout the 1970s. For 40 years it was the definitive theorem on this subject.
Browder’s theorem and its impact (continued)

Browder’s theorem says that there is a framed manifold with nontrivial Kervaire invariant in dimension $2^{j+1} - 2$ iff a certain element in the Adams spectral sequence survives.

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Background and history

Mark Mahowald
Some homotopy theorists, most notably Mahowald, speculated about what would happen if $\theta_j$ existed for all $j$. He derived numerous consequences about homotopy groups of spheres.

The possible nonexistence of the $\theta_j$ for large $j$ was known as the Doomsday Hypothesis.
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There were numerous attempts to construct such manifolds throughout that decade. They all failed. We know now that they failed for good reason. After 1980 the problem faded into the background because it was thought to be too hard.

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London, Ontario 1981
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Vic Snaith and Bill Browder in 1981
Photo by Clarence Wilkerson
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Fast forward to 2009

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In 2009, Snaith published the book "Stable Homotopy Around the Arf-Kervaire Invariant." He wrote this book to "stem the tide of oblivion."
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“In the light of the above conjecture and the failure over fifty years to construct framed manifolds of Arf-Kervaire invariant one this might turn out to be a book about things which do not exist. This [is] why the quotations which preface each chapter contain a preponderance of utterances from the pen of Lewis Carroll.”
Pontryagin’s early work on homotopy groups of spheres

Back to the 1930s
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Lev Pontryagin 1908-1988
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- Pick a regular value $y \in S^k$. Its inverse image will be a smooth $n$-manifold $M$ in $S^{n+k}$.
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.
Pontryagin’s early work on homotopy groups of spheres (continued)

Let $D^k$ be the closure of an open ball around a regular value $y \in S^k$. 

\[ M^n \times D^k \quad \overset{f}{\longrightarrow} \quad S^{n+k} \quad \overset{\phi(M)}{\longrightarrow} \quad \{y\} \]
Let $D^k$ be the closure of an open ball around a regular value $y \in S^k$. If it is sufficiently small, then $V^{n+k} = f^{-1}(D^k) \subset S^{n+k}$ is an $(n+k)$-manifold homeomorphic to $M \times D^k$. 

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A local coordinate system around around the point $y \in S^k$ pulls back to one around $M$ called a framing.
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A local coordinate system around around the point $y \in S^k$ pulls back to one around $M$ called a framing.

There is a way to reverse this procedure. A framed manifold $M^n \subset S^{n+k}$ determines a map $f : S^{n+k} \rightarrow S^k$. 

\[
\begin{align*}
M^n \times D^k &\cong V^{n+k} \rightarrow D^k \\
S^{n+k} &\rightarrow S^k
\end{align*}
\]
Pontryagin’s early work (continued)

Suppose there is homotopy \( h : S^{n+k} \times [0, 1] \to S^k \) between two such maps \( f_1, f_2 : S^{n+k} \to S^k \).
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Pontryagin (1930’s)

Framed cobordism
Let $\Omega_{n,k}^{fr}$ denote the cobordism group of framed $n$-manifolds in $\mathbb{R}^{n+k}$, or equivalently in $S^{n+k}$. 

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Both groups are known to be independent of $k$ for $k > n$. 

Let $\Omega^f_{n,k}$ denote the cobordism group of framed $n$-manifolds in $\mathbb{R}^{n+k}$, or equivalently in $S^{n+k}$. Pontryagin’s construction leads to a homomorphism

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Both groups are known to be independent of $k$ for $k > n$. We denote the resulting stable groups by simply $\Omega^f_n$ and $\pi^S_n$. 

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Let $\Omega^fr_{n,k}$ denote the cobordism group of framed $n$-manifolds in $\mathbb{R}^{n+k}$, or equivalently in $S^{n+k}$. Pontryagin’s construction leads to a homomorphism

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The determination of the stable homotopy groups $\pi^S_n$ is an ongoing problem in algebraic topology.
The Kervaire-Milnor classification of exotic spheres

Into the 60s again
About 50 years ago three papers appeared that revolutionized algebraic and differential topology.

Into the 60s again

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John Milnor’s *On manifolds homeomorphic to the 7-sphere*, 1956. He constructed the first “exotic spheres”, manifolds homeomorphic but not diffeomorphic to the standard $S^7$. They were certain $S^3$-bundles over $S^4$.

- John Milnor
The Kervaire-Milnor classification of exotic spheres (continued)

Michel Kervaire 1927-2007

Michel Kervaire's *A manifold which does not admit any differentiable structure*, 1960.

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The Kervaire-Milnor classification of exotic spheres (continued)

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Michel Kervaire’s *A manifold which does not admit any differentiable structure*, 1960. His manifold was 10-dimensional. I will say more about it later.
The Kervaire-Milnor classification of exotic spheres (continued)

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For example, for \( n = 1, 2, 3, \ldots, 18 \), it will be shown that the order of the group \( \Theta_n \) is respectively:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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The Kervaire-Milnor classification of exotic spheres (continued)


For example, for \( n = 1, 2, 3, \ldots, 18 \), it will be shown that the order of the group \( \oplus_n \) is respectively:

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The Kervaire-Milnor classification of exotic spheres (continued)


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Exotic spheres as framed manifolds

Following Kervaire-Milnor, let $\Theta_n$ denote the group of diffeomorphism classes of exotic $n$-spheres $\Sigma^n$. 
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Two framings of an exotic sphere $\Sigma^n \subset S^{n+k}$ differ by a map to the special orthogonal group $SO(k)$, and this map does not depend on the differentiable structure on $\Sigma^n$. 
Exotic spheres as framed manifolds (continued)

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Heinz Hopf
1894-1971

George Whitehead
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Exotic spheres as framed manifolds (continued)

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They denote the kernel of \( p \) by \( bP_{n+1} \), the group of exotic \( n \)-spheres bounding parallelizable \((n + 1)\)-manifolds.
Exotic spheres as framed manifolds (continued)

Hence we have an exact sequence

\[ 0 \rightarrow bP_{n+1} \rightarrow \Theta_n \overset{p}{\rightarrow} \pi_{n}^S / \text{Im } J. \]
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- *The homomorphism* $p$ *above is onto except possibly when* $n = 4m + 2$ *for* $m \in \mathbb{Z}$, *and then the cokernel has order at most 2.*
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We now know the value of \( bP_{4m+2} \) in every case except \( m = 31 \).
Exotic spheres as framed manifolds (continued)

In other words have a 4-term exact sequence

\[ 0 \rightarrow \Theta_{4m+2} \xrightarrow{\rho} \pi_{4m+2}^S/\text{Im } J \rightarrow \mathbb{Z}/2 \rightarrow bP_{4m+2} \rightarrow 0 \]
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To say more about this we need to define the Kervaire invariant of a framed manifold.
The Arf invariant of a quadratic form in characteristic 2

Back to the 1940s

Let $\lambda$ be a nonsingular anti-symmetric bilinear form on a free abelian group $H$ of rank $2n$ with mod 2 reduction $H$. It is known that $H$ has a basis of the form $\{a_i, b_i : 1 \leq i \leq n\}$ with $\lambda(a_i, a'_i) = 0$, $\lambda(b_j, b'_j) = 0$ and $\lambda(a_i, b_j) = \delta_{i,j}$. 
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Cahit Arf 1910-1997
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Let $\lambda$ be a nonsingular anti-symmetric bilinear form on a free abelian group $H$ of rank $2n$ with mod 2 reduction $\overline{H}$. 

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Browder's work on the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

Background and history
Browder's theorem and its impact
Some early homotopy theory
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Exotic spheres as framed manifolds

The Arf-Kervaire invariant
The Arf invariant
The Kervaire invariant
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$$
The Arf invariant of a quadratic form in characteristic 2 (continued)

In other words, \( \overline{H} \) has a basis for which the bilinear form’s matrix has the symplectic form

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 1 \\
1 & 0 \\
\vdots & \vdots \\
0 & 1 \\
1 & 0
\end{bmatrix}.
\]
The Arf invariant of a quadratic form in characteristic 2 (continued)

A quadratic refinement of $\lambda$ is a map $q : H \to \mathbb{Z}/2$ satisfying

$$q(x + y) = q(x) + q(y) + \lambda(x, y).$$

In 1941 Arf proved that this invariant (along with the number $n$) determines the isomorphism type of $q$. 

1.27
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Money talks: Arf’s definition republished in 2009

Cahit Arf 1910-1997
Bill’s election year definition of the Arf invariant (1968)

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America is a democracy.
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*America is a democracy. If this is not an invariant,*
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America is a democracy. If this is not an invariant, then I don’t know what is.
The Kervaire invariant of a framed \((4m + 2)\)-manifold

Into the 60s
a third time
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Let \(M\) be a \(2m\)-connected smooth closed framed manifold of dimension \(4m + 2\).

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Let \(M\) be a \(2m\)-connected smooth closed framed manifold of dimension \(4m + 2\). Let \(H = H_{2m+1}(M; \mathbb{Z})\), the homology group in the middle dimension. Each \(x \in H\) is represented by an embedding \(i_x : S^{2m+1} \hookrightarrow M\) with a stably trivialized normal bundle.

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Here is a simple example.
The Kervaire invariant of a framed \((4m + 2)\)-manifold

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Here is a simple example. Let \(M = T^2\), the torus, be embedded in \(S^3\) with a framing.
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The Kervaire invariant of a framed \((4m + 2)\)-manifold (continued)

For \(M = T^2 \subset S^3\) and \(x \in H_1(T^2; \mathbb{Z}/2)\), \(q(x)\) is the number of full twists in a cylinder \(V\) neighboring a curve representing \(x\).
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\begin{align*}
\text{Kervaire-Milnor Theorem (1963)} \\
\text{bP}_{4m+2} &= 0 \\
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Kervaire defined a quadratic refinement $q$ on its mod 2 reduction $H$ in terms of each sphere's normal bundle. The Kervaire invariant $\Phi(M)$ is defined to be the Arf invariant of $q$. Recall the Kervaire-Milnor 4-term exact sequence

$$0 \rightarrow \Theta_{4m+2} \rightarrow \pi_{S}^{4m+2}/\text{Im} J \rightarrow \mathbb{Z}/2 \rightarrow bP_{4m+2} \rightarrow 0$$

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$bP_{4m+2} = 0$ iff there is a smooth framed $(4m+2)$-manifold $M$ with $\Phi(M)$ nontrivial.
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Some theorems about $\phi(M)^{4m+2}$ (continued)

Kervaire (1960) showed it must vanish when $m = 2$, so $bP_{10} = \mathbb{Z}/2$. 
Some theorems about $\phi(M)^{4m+2}$ (continued)

Kervaire (1960) showed it must vanish when $m = 2$, so $bP_{10} = \mathbb{Z}/2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure.
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Kervaire (1960) showed it must vanish when $m = 2$, so $bP_{10} = \mathbb{Z}/2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure.

This construction generalizes to higher $m$, but Kervaire’s proof that the boundary is exotic does not.
Brown-Peterson (1966) showed that it vanishes for all positive even $m$. 
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Browder’s theorem

**Browder’s Theorem (1969)**

The Kervaire invariant of a smooth framed \((4m + 2)\)-manifold \(M\) can be nontrivial only if \(m = 2^{j-1} - 1\) for some \(j > 0\). This happens iff the element \(h_2^j\) is a permanent cycle in the Adams spectral sequence.
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Recall that the Kervaire invariant associated with a framing \(F\) is defined in terms of a quadratic map

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H^{2m+1} M = H^{2m+1}(M; \mathbb{Z}/2) \xrightarrow{\psi} \mathbb{Z}/2
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which Browder interprets this as follows. An element in \(H^n X\) is the same thing as a map from \(X\) to the Eilenberg-Mac Lane space

\[
K_n = K(\mathbb{Z}/2, n).
\]
A sketch of Browder’s proof

Now consider the diagram

\[ \begin{array}{ccc}
F_{2m+2} & \xrightarrow{i} & \Sigma K_{2m+1} \\
\downarrow & \searrow & \downarrow i \\
K_{2m+2} & \xrightarrow{Sq^{2m+2}} & K_{4m+4}
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Here the map \(i\) is adjoint to the equivalence \(K_{2m+1} \rightarrow \Omega K_{2m+2}\).
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The space \( F_{2m+2} \) has two nontrivial homotopy groups,

\[
\pi_n F_{2m+2} = \begin{cases} 
\mathbb{Z}/2 & \text{for } n = 2m + 2 \\
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The map \( \hat{i} \) is an equivalence thru dimension \( 4m + 3 \) and

\[
\pi_{4m+2+k} \Sigma^k K_{2m+1} = \mathbb{Z}/2 \quad \text{for } k > 0.
\]
A sketch of Browder’s proof (continued)

A framed embedding of $M$ in $\mathbb{R}^{k+4m+2}$ and a class $x \in H^{2m+1}M$ yields a diagram
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where the Pontryagin map $p_F$ depends on the choice of framing $F$. 

Browder's strategy: Find the most general possible and simplest situation in which the Kervaire element can be defined and then study the place of framed manifolds in this situation.
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Given a vector bundle \( \xi \) over a space \( X \), let \( w(\xi) \) denote its total Stiefel-Whitney class

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Hence $v_n(\xi)$ for each $n > 0$ is a certain polynomial in the Stiefel-Whitney classes.
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For a \((4m + 2)\)-manifold \(M\) we define \(\nu_i(M) \in H^i M\) to be the \(i\)th Wu class of its normal bundle.
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For a \((4m + 2)\)-manifold \(M\) we define \(\nu_i(M) \in H^iM\) to be the \(i\)th Wu class of its normal bundle. It is known that for \(x \in H^{4m+2-i}M\),

\[ Sq^i x = \nu_i x \in H^{4m+2}M. \]

This implies via Poincaré duality that \(\nu_i(M) = 0\) for \(i > 2m + 1\).
Wu orientations

\[ \nu(\xi) = (Sq^{-1}w(\xi))^{-1}. \]

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\[ Sq^i x = \nu_i x \in H^{4m+2}M. \]

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Consider the diagram

\[
\begin{array}{ccc}
B\langle v_{2m+2} \rangle & \xrightarrow{\pi} & BO \\
\xrightarrow{\nu} & & \xrightarrow{v_{2m+2}} \xrightarrow{\ast} K_{2m+2} \\
\end{array}
\]
Wu orientations

\[ \nu(\xi) = (\text{Sq}^{-1} w(\xi))^{-1}. \]

For a \((4m + 2)\)-manifold \(M\) we define \(v_i(M) \in H^i M\) to be the \(i\)th Wu class of its normal bundle. It is known that for \(x \in H^{4m+2-i} M\),

\[ \text{Sq}^i x = v_i x \in H^{4m+2} M. \]

This implies via Poincaré duality that \(v_i(M) = 0\) for \(i > 2m + 1\).

Consider the diagram

\[ \begin{array}{ccc}
  M & \xrightarrow{\nu} & \nu \\
  \downarrow & & \downarrow \pi \\
  B\langle v_{2m+2} \rangle & \xrightarrow{\pi} & BO & \xrightarrow{v_{2m+2}} & K_{2m+2} \\
\end{array} \]

where \(BO\) is the classifying space of the stable orthogonal group \(O\),
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Consider the diagram

\[
\begin{array}{ccc}
B\langle v_{2m+2} \rangle & \xrightarrow{\pi} & BO \\
\xrightarrow{\nu} & \xrightarrow{\nu} & \xrightarrow{v_{2m+2}} K_{2m+2}
\end{array}
\]

where \(BO\) is the classifying space of the stable orthogonal group \(O\), \(\nu\) is the map inducing the normal bundle,
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\[ \nu(\xi) = \left( Sq^{-1} w(\xi) \right)^{-1}. \]

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Consider the diagram

\[
\begin{array}{ccc}
B\langle v_{2m+2} \rangle & \xrightarrow{\pi} & BO \\
\downarrow & & \downarrow \nu \\
\hat{\nu} & & * \\
& M & \xrightarrow{v_{2m+2}} K_{2m+2}
\end{array}
\]

where \(BO\) is the classifying space of the stable orthogonal group \(O\), \(\nu\) is the map inducing the normal bundle, and \(B\langle v_{2m+2} \rangle\) is the fiber of the map \(v_{2m+2}\).
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Consider the diagram

\[ \begin{array}{c}
B\langle v_{2m+2} \rangle \\
\pi \downarrow \quad \nu \downarrow \\
BO \\
\nu v_{2m+2} \quad \rightarrow \quad K_{2m+2}
\end{array} \]

where \(BO\) is the classifying space of the stable orthogonal group \(O\), \(\nu\) is the map inducing the normal bundle, and \(B\langle v_{2m+2} \rangle\) is the fiber of the map \(v_{2m+2}\). Then the composite \(v_{2m+2} \cdot \nu\) is null so the indicated lifting exists, but not uniquely.
Wu orientations

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For a \((4m+2)\)-manifold \(M\) we define \(v_i(M) \in H^i M\) to be the \(i\)th Wu class of its normal bundle. It is known that for \(x \in H^{4m+2-i} M\),
\[ Sq^i x = v_i x \in H^{4m+2} M. \]

This implies via Poincaré duality that \(v_i(M) = 0\) for \(i > 2m + 1\).

Consider the diagram

\[
\begin{array}{ccc}
    B\langle v_{2m+2} \rangle & \xrightarrow{\pi} & BO \\
    & \downarrow{\nu} & \downarrow{\nu} \\
    & \overset{\nu}{\Rightarrow} B\langle v_{2m+2} \rangle & \rightarrow K_{2m+2}
\end{array}
\]

where \(BO\) is the classifying space of the stable orthogonal group \(O\), \(\nu\) is the map inducing the normal bundle, and
\(B\langle v_{2m+2} \rangle\) is the fiber of the map \(v_{2m+2}\). Then the composite \(v_{2m+2} \cdot \nu\) is null so the indicated lifting exists, but not uniquely. Browder calls \(\nu\) a Wu orientation of \(M\).
The Browder spectrum

\[ K_{2m+1} \rightarrow B\langle v_{2m+2} \rangle \rightarrow BO \rightarrow K_{2m+2} \]
The Browder spectrum

We now consider the Thom spectra associated the universal bundle over $BO$ and its pullbacks.
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We now consider the Thom spectra associated the universal bundle over $BO$ and its pullbacks. The diagram becomes

$$
\begin{align*}
K_{2m+1} & \longrightarrow B\langle v_{2m+2} \rangle \quad \pi \quad BO \quad \nu \quad v_{2m+2} \quad K_{2m+2} \\
\end{align*}
$$

$K_{2m+1}$ $\xrightarrow{\nu}$ $B\langle v_{2m+2} \rangle$ $\xrightarrow{\pi}$ $BO$ $\xrightarrow{\nu}$ $v_{2m+2}$ $\xrightarrow{}$ $K_{2m+2}$

$K_{2m+1}$ $\xrightarrow{\nu}$ $Br_{2m+2}$ $\xrightarrow{T\nu}$ $MO$

$K_{2m+1}$ $\xrightarrow{\nu}$ $B\langle v_{2m+2} \rangle$ $\xrightarrow{\pi}$ $BO$ $\xrightarrow{\nu}$ $v_{2m+2}$ $\xrightarrow{}$ $K_{2m+2}$

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The Browder spectrum

We now consider the Thom spectra associated the universal bundle over $BO$ and its pullbacks. The diagram becomes

$$
\begin{array}{c}
K_{2m+1} \xrightarrow{\nu} B\langle v_{2m+2} \rangle \xrightarrow{\pi} BO \xrightarrow{v_{2m+2}} K_{2m+2} \\
\end{array}
$$

where $T(\nu_M)$ is the Thom spectrum for the normal bundle of $M$, and

$$
\begin{array}{c}
K_{2m+1} \xrightarrow{\bar{\nu}} Br_{2m+2} \xrightarrow{\bar{\nu}} MO \\
\end{array}
$$
The Browder spectrum

\[ \begin{array}{c}
K_{2m+1} \rightarrow B\langle v_{2m+2} \rangle \rightarrow BO \rightarrow K_{2m+2} \\
\end{array} \]

We now consider the Thom spectra associated the universal bundle over \( BO \) and its pullbacks. The diagram becomes

\[ \begin{array}{c}
K_{2m+1} \rightarrow Br_{2m+2} \rightarrow MO \\
\end{array} \]

where \( T(\nu_M) \) is the Thom spectrum for the normal bundle of \( M \), \( K_{2m+1} \) here denotes the suspension spectrum of the space \( K_{2m+1} \)
The Browder spectrum

\[ K_{2m+1} \rightarrow B\langle v_{2m+2} \rangle \rightarrow BO \rightarrow K_{2m+2} \]

We now consider the Thom spectra associated the universal bundle over \( BO \) and its pullbacks. The diagram becomes

\[ K_{2m+1} \rightarrow Br_{2m+2} \rightarrow MO \]

where \( T(\nu_M) \) is the Thom spectrum for the normal bundle of \( M \), and \( K_{2m+1} \) denotes the suspension spectrum of the space \( K_{2m+1} \) and \( Br_{2m+2} \), the \( m \)th Browder spectrum, is the Thom spectrum associated with \( B\langle v_{2m+2} \rangle \).
The Browder spectrum (continued)

\[ \Sigma^\infty K_{2m+1} \rightarrow \text{Br}_{2m+2} \rightarrow T(\nu_M) \]

The Spanier-Whitehead dual of \( T(\nu_M) \) is \( \Sigma^{-4m-2} M \), so we have a map

\[ \text{D} \text{Br}_{2m+2} \eta \rightarrow \Sigma^{-4m-2} M. \]

Both of these spectra have no cells in positive dimensions and \( \text{Sq}^{2m+2} \) maps trivially to \( H^0 \).

Now suppose we have an element \( x \in H_{2m+1} M \) with \( \eta^*(x) = 0 \).

Stably we have

\[ \text{D} \text{Br}_{2m+2} \eta \rightarrow \rightarrow \Sigma^{-4m-2} K_{2m+1} \rightarrow X. \]
The Browder spectrum (continued)

\[ \Sigma^\infty K_{2m+1} \rightarrow \text{Br}_{2m+2} \rightarrow T(\nu_M) \rightarrow \text{MO} \]

The Spanier-Whitehead dual of \( T(\nu_M) \) is \( \Sigma^{−4m−2}M_+ \), so we have a map

\[ D\text{Br}_{2m+2} \rightarrow \Sigma^{−4m−2}M_+ \]
The Browder spectrum (continued)

\[
\begin{array}{ccc}
\Sigma^\infty K_{2m+1} & \longrightarrow & \text{Br}_{2m+2} \\
& \text{p} \downarrow & \text{Br} \downarrow \\
& \Sigma \text{K} \rightarrow \text{MO}
\end{array}
\]

The Spanier-Whitehead dual of \( T(\nu_M) \) is \( \Sigma^{-4m-2} M_+ \), so we have a map

\[
\text{DBr}_{2m+2} \overset{\eta}{\longrightarrow} \Sigma^{-4m-2} M_+.
\]

Both of these spectra have no cells in positive dimensions and \( Sq^{2m+2} \) maps trivially to \( H^0 \).
The Browder spectrum (continued)

\[
\begin{array}{c}
\Sigma^\infty K_{2m+1} \rightarrow Br_{2m+2} \rightarrow MO \\
\end{array}
\]

The Spanier-Whitehead dual of \( T(\nu_M) \) is \( \Sigma^{-4m-2}M_+ \), so we have a map

\[
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\]

Both of these spectra have no cells in positive dimensions and \( Sq^{2m+2} \) maps trivially to \( H^0 \). Now suppose we have an element \( x \in H^{2m+1}M \) with \( \eta^*(x) = 0 \).
The Browder spectrum (continued)

\[
\begin{array}{ccc}
\Sigma^\infty K_{2m+1} & \xrightarrow{\bar{p}} & \text{Br}_{2m+2} \\
\downarrow & & \downarrow T \nu \\
\downarrow T \nu & & \downarrow T \nu \\
\Sigma & \xrightarrow{\eta} & \Sigma^{-4m-2} \mathbb{M}_+ \\
\end{array}
\]

The Spanier-Whitehead dual of \( T(\nu_M) \) is \( \Sigma^{-4m-2} \mathbb{M}_+ \), so we have a map

\[
DBr_{2m+2} \xrightarrow{\eta} \Sigma^{-4m-2} \mathbb{M}_+.
\]

Both of these spectra have no cells in positive dimensions and \( Sq^{2m+2} \) maps trivially to \( H^0 \). Now suppose we have an element \( x \in H^{2m+1} \mathbb{M} \) with \( \eta^*(x) = 0 \). Stably we have

\[
DBr_{2m+2} \xrightarrow{\eta} \Sigma^{-4m-2} \mathbb{M}_+ \\
\xrightarrow{g} \Sigma^{-4m-2} K_{2m+1} \xrightarrow{x} K
\]
Let $q = 2m + 1$, so our diagram reads

$$DBr_{q+1} \xrightarrow{\eta} \Sigma^{-2q} M_+$$

$$\xymatrix{ X \ar[r]^g & \Sigma^{-2q} K_q \ar[r]^\chi & K}$$
The Browder spectrum (continued)

Let \( q = 2m + 1 \), so our diagram reads

\[
DBr_{q+1} \xrightarrow{\eta} \Sigma^{-2q} M_+ \\
\| \downarrow \downarrow x \\
X \xrightarrow{g} \Sigma^{-2q} K_q \longrightarrow K
\]

Consider the following diagram with exact rows in black:

\[
\begin{array}{cccccccc}
0 & & \xleftarrow{\lambda q} & & \xleftarrow{\alpha} & & 0 \\
H^{-q} X & & \xleftarrow{g^*} & & H^{-q} K & & \xleftarrow{H^{-q}(K, X)} & & H^{-1-q} X \\
\downarrow Sq^{q+1} & & \downarrow Sq^{q+1} & & \downarrow 0 & & & & & & & & & & & & & & & & \\
H^1 K & & \xleftarrow{H^1(K, X)} & & H^0 X & & \xleftarrow{0} & & H^0 K \\
\downarrow & & \downarrow & & \downarrow & & 0 & & \downarrow Sq^q \lambda q & & 0 & & \downarrow Sq^{q+1} \alpha & & \psi(x)
\end{array}
\]

Browder shows that the operation \( \psi(x) \) is quadratic.
The Browder spectrum (continued)

Let \( q = 2m + 1 \), so our diagram reads

\[
DBr_{q+1} \xrightarrow{\eta} \Sigma^{-2q} M_+ \xrightarrow{x} \Sigma^{-2q} K_q \xrightarrow{m} K
\]

Consider the following diagram with exact rows in black:

\[
\begin{array}{c}
0 \xleftarrow{\iota q} \overset{\lambda q}{\rightarrow} H^{-q} X \xleftarrow{\alpha} H^{-q} K \\
\downarrow \text{Sq}^{q+1} & \downarrow \text{Sq}^{q+1} & \downarrow 0 \\
H^1 K & H^1(K, X) & H^0 X \xleftarrow{0} H^0 K
\end{array}
\]

The diagram chase is shown in red.
The Browder spectrum (continued)

Let \( q = 2m + 1 \), so our diagram reads

\[
\begin{array}{c}
DBr_{q+1} \xrightarrow{\eta} \Sigma^{-2q}M_+ \\
\| \\
X \xrightarrow{g} \Sigma^{-2q}K_q \xrightarrow{\alpha} K
\end{array}
\]

Consider the following diagram with exact rows in black:

\[
\begin{array}{c}
0 \xleftarrow{\iota_q} H^{-q}X \xleftarrow{g^*} H^{-q}K \xleftarrow{H^{-q}(K, X)} H^{-1-q}X \\
\downarrow Sq^{q+1} \downarrow Sq^{q+1} \downarrow 0 \\
H^1K \xleftarrow{H^1(K, X)} H^0X \xleftarrow{0} H^0K
\end{array}
\]

The diagram chase is shown in red. The element \( \psi(x) \) is independent of the choice of \( \alpha \).
The Browder spectrum (continued)

Let \( q = 2m + 1 \), so our diagram reads

\[
\begin{array}{c}
DBr_{q+1} \xrightarrow{\eta} \Sigma^{-2q}M_+ \\
\| \\
X \xrightarrow{g} \Sigma^{-2q}K_q = K
\end{array}
\]

Consider the following diagram with exact rows in black:

\[
\begin{array}{c}
0 \leftarrow \iota q \leftarrow \alpha \\
H^{-q}X \xleftarrow{g^*} H^{-q}K \leftarrow H^{-q}(K, X) \leftarrow H^{-1-q}X \\
\downarrow Sq^{q+1} \downarrow Sq^{q+1} \downarrow 0 \\
H^1K \leftarrow H^1(K, X) \leftarrow H^0X \leftarrow 0 \leftarrow H^0K
\end{array}
\]

The diagram chase is shown in red. The element \( \psi(x) \) is independent of the choice of \( \alpha \). Browder shows that the operation \( \psi \) is quadratic.
The Browder spectrum (continued)

If the manifold $M$ has a framing $F$ we get
The Browder spectrum (continued)

If the manifold $M$ has a framing $F$ we get

$$
\sum_\infty K_{2m+1} \xrightarrow{\overline{p}} \text{Br}_{2m+2} \xrightarrow{T(\nu_M)} \text{MO}
$$
The Browder spectrum (continued)

If the manifold $M$ has a framing $F$ we get

$$
\begin{array}{c}
\Sigma^\infty K_{2m+1} \\
\xrightarrow{p} \\
\xrightarrow{\bar{p}}
\end{array}
\xrightarrow{T(\nu_M)}
\xrightarrow{T_\nu}
\xrightarrow{T_\nu}
\xrightarrow{T_\nu}
Br_{2m+2} \xrightarrow{\psi} MO

This means we can replace $X = DBr_{2m+2}$ by $S^0$, so the next diagram becomes

$$
\begin{array}{c}
S^0 \\
\xrightarrow{p_F} \\
\xrightarrow{x}
\end{array}
\xrightarrow{\Sigma^{-4m-2} M \_+}
\xrightarrow{\Sigma^{-4m-2} K_{2m+1}}
\xrightarrow{\Sigma^{-4m-2} K_{2m+1}}
$$
The Browder spectrum (continued)

If the manifold $M$ has a framing $F$ we get

$$
\begin{array}{c}
\Sigma^\infty K_{2m+1} \\
\xrightarrow{\psi} \\
\xrightarrow{\nu} \\
\xrightarrow{T} \\
\xrightarrow{T \nu}
\end{array}
\quad
\begin{array}{c}
\xrightarrow{\nu M} \\
\xrightarrow{\nu} \\
\xrightarrow{T \nu}
\end{array}
\quad
\begin{array}{c}
S^0 \\
\xrightarrow{T} \\
\xrightarrow{T \nu}
\end{array}
\quad
\begin{array}{c}
\xrightarrow{T \nu}
\end{array}
\quad
\begin{array}{c}
\xrightarrow{T \nu}
\end{array}
\quad
\begin{array}{c}
\xrightarrow{T \nu}
\end{array}

This means we can replace $X = D\text{Br}_{2m+2}$ by $S^0$, so the next diagram becomes

$$
\begin{array}{c}
S^0 \\
\xrightarrow{p_F} \\
\xrightarrow{\phi} \\
\xrightarrow{x}
\end{array}
\quad
\begin{array}{c}
\xrightarrow{\psi} \\
\xrightarrow{\phi M} \\
\xrightarrow{x}
\end{array}
\quad
\begin{array}{c}
\xrightarrow{\psi} \\
\xrightarrow{\phi K_{2m+1}}
\end{array}

This is Browder's interpretation of the quadratic operation $\psi$ described earlier.
The homotopy type of $\text{Br}_{2m+2}$

A framed $(4m + 2)$-manifold $M$ with nontrivial Kervaire invariant represents, via Pontryagin’s isomorphism, a nontrivial map
The homotopy type of $\text{Br}_{2m+2}$

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$$S^{4m+2} \xrightarrow{\theta} S^0.$$
The homotopy type of $\text{Br}_{2m+2}$

A framed $(4m + 2)$-manifold $M$ with nontrivial Kervaire invariant represents, via Pontryagin’s isomorphism, a nontrivial map

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Browder shows that the composite map to the Browder spectrum

$$S^{4m+2} \xrightarrow{\theta} S^0 \xrightarrow{} \text{Br}_{2m+2}$$

must also be nontrivial.
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He analyzes the homotopy type of $\text{Br}_{2m+2}$ and gets a diagram
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$$S^{4m+2} \xrightarrow{\theta} S^0 \longrightarrow \text{Br}_{2m+2}$$

must also be nontrivial.

He analyzes the homotopy type of $\text{Br}_{2m+2}$ and gets a diagram

$$\text{Br}_{2m+2} \leftarrow \text{Br}_{2m+2}^{(1)} \leftarrow \text{Br}_{2m+2}^{(2)} \leftarrow \binom{(4m + 2) \text{-connected}}{\text{fiber}}$$

$\bar{p}$

$\bar{q}$

$\bar{h}$

$\bar{k}$

$MO \quad K_{2m+1} \wedge MO \quad K_{4m+2}$
The homotopy type of $\text{Br}_{2m+2}$ (continued)

\[
\begin{array}{cccc}
S^0 & \xrightarrow{\theta} & S^{4m+2} \\
\downarrow & & \downarrow \\
\text{Br}_{2m+2} & \xleftarrow{\bar{p}} & \text{Br}_{2m+2}^{(1)} & \xleftarrow{h} \text{Br}_{2m+2}^{(2)} & \xleftarrow{k} \left( (4m + 2)\text{-connected fiber} \right) \\
\downarrow & & \downarrow & \downarrow \\
MO & K_{2m+1} \wedge MO & K_{4m+2} \\
\end{array}
\]
The homotopy type of $\text{Br}_{2m+2}$ (continued)

$$
\begin{align*}
S^0 & \leftarrow \theta \rightarrow S^{4m+2} \\
\Downarrow & \quad \Downarrow \\
\text{Br}_{2m+2} & \leftarrow \text{Br}_{2m+2}^{(1)} \leftarrow \text{Br}_{2m+2}^{(2)} \leftarrow (4m+2)\text{-connected fiber} \\
\Downarrow & \quad \Downarrow & \quad \Downarrow \\
MO & \leftarrow K_{2m+1} \wedge MO & K_{4m+2}
\end{align*}
$$

Here each horizontal map is the inclusion of the fiber of the following vertical map.

We know that $MO$ is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra. This means that $\text{Br}_{2m+2}$ is a 3-stage Postnikov system in the relevant range of dimensions. It follows that $\theta$ must be detected by an element on the 2-line of the Adams spectral sequence. An explicit description of the map $k$ rules out all elements other than $h_{2j}$, which is shown to detect the Kervaire invariant in dimension $2j+1-2$. This completes the proof of the theorem.
The homotopy type of $\text{Br}_{2m+2}$ (continued)

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The homotopy type of $\text{Br}_{2m+2}$ (continued)

$S^0 \xleftarrow{\theta} S^{4m+2}$

$\text{Br}_{2m+2} \xleftarrow{\theta} \text{Br}_{2m+2}^{(1)} \xleftarrow{\theta} \text{Br}_{2m+2}^{(2)} \leftarrow \left(4m+2\right)$-connected fiber

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The homotopy type of $\text{Br}_{2m+2}$ (continued)

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It follows that $\theta$ must be detected by an element on the 2-line of the Adams spectral sequence. An explicit description of the map $k$ rules out all elements other than $h^2_j$, which is shown to detect the Kervaire invariant in dimension $2^{j+1} - 2$. 

$$
\begin{array}{cccc}
S^0 & \xleftarrow{\theta} & S^{4m+2} \\
\downarrow & & \downarrow \\
\text{Br}_{2m+2} & \xleftarrow{\bar{p}} & \text{Br}_{2m+2}^{(1)} & \xleftarrow{h} & \text{Br}_{2m+2}^{(2)} & \xleftarrow{k} & (4m+2)\text{-connected fiber} \\
MO & \downarrow & K_{2m+1} \wedge MO & \downarrow & K_{4m+2} \\
\end{array}
$$
The homotopy type of $Br_{2m+2}$ (continued)

$$S^0 \xleftarrow{\theta} S^{4m+2}$$

$Br_{2m+2} \xleftarrow{\bar{p}} Br_{2m+2}^{(1)} \xleftarrow{h} Br_{2m+2}^{(2)} \xleftarrow{k} \left( (4m + 2)\text{-connected fiber} \right)$

Here each horizontal map is the inclusion of the fiber of the following vertical map. We know that $MO$ is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra. This means that $Br_{2m+2}$ is a 3-stage Postnikov system in the relevant range of dimensions.

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This completes the proof of the theorem.