The $E_2$-term of the Adams–Novikov spectral sequence [2] for a spectrum $X$ localized at the prime $p$ has the form

$$\text{Ext}_{BP_*BP^*} (BP_* , M)$$

(0.1)

where $BP$ is the Brown–Peterson spectrum [2] at $p$ and $M$ is the "$BP_*BP$-comodule" [1] $BP_*(X)$. Recall [2] that $BP_* = \pi_*(BP) = \mathbb{Z}_{(p)}[v_1 , v_2 , \cdots]$, $|v_i| = 2p^i - 2$. The purpose of this paper is to identify (0.1) with an Ext group over a smaller "Hopf algebra" in case $M$ is $v_i$-local, by which we mean that $v_i$ acts on $M$ bijectively.

The first theorem in this direction is due to Jack Morava [14]. Morava shows that if $M$ is a $v_i$-local comodule which is killed by the ideal $I_n = (p, v_1, \cdots, v_{n-1})$ and finitely generated over $v_1^{-1}BP_*/I_n$, then (0.1) may be computed in terms of the continuous cohomology of a certain $p$-adic Lie group with coefficients in a finite dimensional representation over $\mathbb{F}_p^*$ constructed out of $M$.

We prove the following "covariant" analogue of this theorem in Section 2. Let $K(0)_* = Q$, and $K(n)_* = \mathbb{F}_p[v_n , v_n^{-1}]$ for $n > 0$, with the obvious $BP_*$-algebra structures. Let $K(n)_* K(n) = K(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} K(n)_*$; it inherits from $BP_*BP$ the structure of a Hopf algebra over the graded field $K(n)_*$.

**Theorem 2.10.** If $M$ is $v_i$-local and $I_nM = 0$, then

$$\text{Ext}_{BP_*BP^*} (BP_* , M) \cong \text{Ext}_{K(n)_* K(n)} (K(n)_* , K(n)_* \otimes_{BP_*} M)$$

under the natural map.

In Section 3 we strengthen Theorem 2.10 by dropping the requirement that $I_nM = 0$. Let $E(n)_* = \mathbb{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$ with the obvious $BP_*$-algebra structure, and let $E(n)_* E(n) = E(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} E(n)_*$. Then we have

**Theorem 3.10.** If $M$ is $v_i$-local, then

$$\text{Ext}_{BP_*BP^*} (BP_* , M) \cong \text{Ext}_{E(n)_* E(n)} (E(n)_* , E(n)_* \otimes_{BP_*} M)$$

under the natural map.

Thus higher generators can be neglected, at the cost of introducing a rather complicated set of relations into $BP_*BP$.

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These change of rings theorems rest on a version of Shapiro's Lemma which we prove in Section 1. In this section we also review the definition and elementary properties of Hopf algebroids.

This paper forms the link between the work of the second author in [16] and [17] and our joint work with W. S. Wilson in [12] and [13]. The latter has resulted in proofs of the essentiality of a great many elements in the stable homotopy ring. Briefly stated, our program is to compute a portion of the Novikov $E_2$-term for the sphere, $\text{Ext}_{BP,BP}^* (BP_*, BP_*)$, by means of the "Bockstein" long exact sequences induced in $\text{Ext}_{BP,BP}^* (BP_*, )$ by the short exact sequences

$$0 \to BP_*/I_n \to BP_*/I_n \to BP_*/I_{n+1} \to 0.$$  (0.2)

The foundation of the analysis of these long exact sequences is the study of

$$\lim \text{Ext}_{BP,BP}^* (BP_*, BP_*/I_n),$$

where the maps in the directed system are induced by multiplication by $v_n$. It is easy to see that this limit is just $\text{Ext}_{BP,BP}^* (BP_*, v_n^{-1}BP_*/I_n)$, which Theorem 2.10 now identifies with $\text{Ext}_{K(n),K(n)}^* (K(n)_*, K(n)_*)$. This is a substantial simplification: $K(n)_* K(n)$ is a Hopf algebra over a field, while $BP_* BP$ is a (much larger) Hopf algebroid (see Section 1 below) over the (complicated) ring $BP_*$. Thus the computation of $\text{Ext}_{K(n),K(n)}^* (K(n)_*, K(n)_*)$ falls within the scope of the methods of Peter May's thesis, and is carried out in certain cases in [17].

We wish to thank J. F. Adams, Peter Landweber, and Steve Wilson for useful conversations, and we gratefully acknowledge the deep influence Jack Morava has had on our work.

Section 1. Hopf algebroids and a Shapiro's Lemma.

We shall work here in somewhat greater generality than is necessary in the sequel. We begin by recalling Adams' algebraic context [1] for cooperation coalgebras for generalized homology theories.

Let $K$ be a commutative ring. A Hopf algebroid $(A, \Gamma)$ (over $K$) is a co-groupoid object in the category of commutative graded $K$-algebras. Thus we have structural $K$-algebra maps $\eta_R : A \to \Gamma$ (source, target), $\epsilon : \Gamma \to A$ (identity), $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$ (composition), $\iota : \Gamma \to \Gamma$ (inverse). Here $\Gamma$ becomes a left $A$-module via $\eta_L$ and a right $A$-module via $\eta_R$, and $\Gamma \otimes_A \Gamma$ is the usual tensor product of bimodules. We require of these maps that $\Delta$ and $\epsilon$ be $A$-bimodule maps and that the following diagrams commute.

$$\begin{array}{ccc}
A \otimes_A \Gamma & \xleftarrow{\sim} & \Gamma \\
\epsilon \otimes_A \Gamma & \downarrow{\Delta} & \Gamma \otimes_A \epsilon \\
\Gamma \otimes_A A & \xrightarrow{\sim} & \Gamma \end{array}$$
where $c \cdot \Gamma(\gamma_1 \otimes \gamma_2) = c(\gamma_1)\gamma_2$ and $\Gamma \cdot c(\gamma_1 \otimes \gamma_2) = \gamma_1c(\gamma_2)$.

We leave to the reader the amusement of interpreting these diagrams as the axioms for a cogroupoid object. We will frequently let the coefficient algebra $A$ be understood and write $\Gamma$ for $(A, \Gamma)$.

A (left) $\Gamma$-comodule is a left $A$-module $M$ together with an $A$-linear map $\psi: M \rightarrow \Gamma \otimes_A M$ such that the following diagrams commute.

$$
\begin{array}{ccc}
M & \xrightarrow{\psi} & \Gamma \otimes_A M \\
\downarrow & \cong & \downarrow \epsilon \otimes_A M \\
A \otimes_A M & \rightarrow & \end{array}
\begin{array}{ccc}
M & \xrightarrow{\psi} & \Gamma \otimes_A M \\
\downarrow & & \downarrow \Delta \otimes_A M \\
\Gamma \otimes_A M & \rightarrow & \end{array}
\begin{array}{ccc}
\Gamma \otimes_A M & \rightarrow & \\
\psi & & \psi \\
0 & \rightarrow & \Gamma \otimes_A L \rightarrow \Gamma \otimes_A M \rightarrow \Gamma \otimes_A N
\end{array}
$$

A morphism of $\Gamma$-comodules is an $A$-linear map $f: M \rightarrow N$ such that $\psi_N = (\Gamma \otimes_A f)\psi_M$.

Right comodules are defined analogously.

In general the category $(\Gamma$-comod) of $\Gamma$-comodules may fail to have kernels; but if $\Gamma$ is $A$-flat (as left or equivalently as right module) then the bottom row of the diagram

$$
0 \rightarrow L \rightarrow M \rightarrow N
$$

A morphism of $\Gamma$-comodules is an $A$-linear map $f: M \rightarrow N$ such that $\psi_N = (\Gamma \otimes_A f)\psi_M$. Right comodules are defined analogously.
is exact if the top row is, so $\psi_L$ exists and defines a comodule structure on $L$. (Γ-comod) is then easily seen to be an Abelian category.

Our central example of a Hopf algebroid is $(BP_*, BP_*, BP)$; see [2], [10]. $BP_*BP = BP_*[t_1, t_2, \cdots]$ is flat over $BP_*$. For any spectrum $X$, $BP_*(X)$ is a $BP_*BP$-comodule.

For another example, note that if $\eta_L = \eta_R$ then $\Gamma$ is just a commutative Hopf algebra over the graded commutative ring $A$.

A $\Gamma$-comodule $M$ is a relative injective iff it is a summand of an extended comodule, i.e., one of the form $\Gamma \otimes_A X$ for some $A$-module $X$. A comodule map is a relative monomorphism iff it is split-mono as a map of $A$-modules. Using these notions we may build resolutions in the usual way, and define $\operatorname{Ext}_r^*(A, \_)$ as the relative right derived functor of the graded $K$-module-valued functor $\operatorname{Hom}_r(A, \_ )$. We refer the reader to [4], section 4, for a discussion of relative homological algebra. Throughout the paper, “injective” will always mean “relative injective” in this sense.

In particular we have the standard [4] or cobar resolution $L^*(\Gamma; M)$. In degree $n$,

$$L^*(\Gamma; M) = \Gamma \otimes_A \cdots \otimes_A \Gamma \otimes_A M$$

with $(n + 1)$ factors of $\Gamma$, and differential

$$d(\gamma_0 \otimes \cdots \otimes \gamma_n \otimes m) = \sum_{i=0}^n (-1)^{r(i)} \gamma_0 \otimes \cdots \otimes \gamma_i' \otimes \gamma_i'' \otimes \cdots \otimes \gamma_n \otimes m + (-1)^{r(n+1)} \sum \gamma_0 \otimes \cdots \otimes \gamma_n \otimes m' \otimes m''$$

where

$$\Delta \gamma_i = \sum \gamma_i' \otimes \gamma_i'', \quad \psi m = \sum m' \otimes m''$$

and

$$\sigma(i) = |\gamma_0| + \cdots + |\gamma_{i-1}| + i.$$  

The usual contracting homotopy

$$s(\gamma_0 \otimes \cdots \gamma_n \otimes m) = \epsilon(\gamma_0) \gamma_1 \otimes \cdots \otimes \gamma_n \otimes m$$

shows that $L^*(\Gamma; M)$ is a relative injective resolution, so that $\operatorname{Ext}_r^*(A, M)$ is the homology of the cobar complex

$$\Omega^*(\Gamma; M) = \operatorname{Hom}_r(A, L^*(\Gamma; M)).$$

Our first job is to show that $\operatorname{Ext}_r^*(A, M)$ may be computed using a wider class of complexes than those described above.

Lem 1.1 Let $\Gamma$ be $A$-flat and let $0 \to M \to I^0 \to I^1 \to \cdots$ be a sequence of $\Gamma$-comodules which is exact (over $K$) and such that for each $i$, $\operatorname{Ext}_r^*(A, I^i) = 0$ for all $q > 0$. Then $\operatorname{Ext}_r^*(A, M) = H(\operatorname{Hom}_r(A, I^*))$. 

Consider the double complex

\[ X^{**} = \Omega^*(\Gamma; I^*). \]

Both of the associated spectral sequences ([3], XV §6) collapse at \( E_2 \) to give the desired equality.

Now let \( \pi = (\pi, \pi) : (A, \Gamma) \to (B, \Sigma) \) be a map of Hopf algebroids. For a \( \Gamma \)-comodule \( M, B \otimes_A M \) is naturally a \( \Sigma \)-comodule via the \( B \)-linear extension of \( \psi \):

\[ M \overset{\psi}{\longrightarrow} \Gamma \otimes_A M \rightarrow \Sigma \otimes_A M. \]

The usual lifting argument shows that \( \pi \) induces a map

\[ \pi_* : \text{Ext}_{\Gamma}^* (A, M) \to \text{Ext}_{\Sigma}^* (B, B \otimes_A M). \]

Symmetrically, if \( M \) is a right \( \Gamma \)-comodule, then \( M \otimes_A B \) is a right \( \Sigma \)-comodule. In particular \( \Gamma \otimes_A B \) is a right \( \Sigma \)-comodule. It is also obviously a left \( \Gamma \)-comodule, and in a compatible way, so we have functors

\[ (\Gamma \text{-comod}) \overset{B \otimes_A}{\longrightarrow} (\Sigma \text{-comod}). \tag{1.2} \]

Here \( N' \square_{\Sigma} N'' \) is the cotensor product of the right \( \Sigma \)-comodule \( N' \) and the left \( \Sigma \)-comodule \( N'' \), defined as the \( K \)-module kernel of

\[ \psi_{N'} \otimes N'' - N' \otimes \psi_{N''} : N' \otimes B N'' \to N' \otimes_B \Sigma \otimes_B N''. \]

Write \( \pi_* = B \otimes_A \) and \( \pi^* = (\Gamma \otimes_A B) \square_{\Sigma} \).

These functors are adjoint:

\[ \text{Hom}_{\Gamma} (M, \pi^* N) \cong \text{Hom}_{\Sigma} (\pi_* M, N) \]

naturally in \( M \in (\Gamma \text{-comod}), N \in (\Sigma \text{-comod}) \). In particular

\[ \text{Hom}_{\Gamma} (A, \pi^* N) \cong \text{Hom}_{\Sigma} (B, N). \]

Since they will be of use later we display the adjunction morphisms [4]. The front adjunction \( \alpha_N : M \to \pi_* \pi^* N \) is defined as the composite of the morphisms in the top line of the commutative diagram

\[ M \overset{\sim}{\longrightarrow} \Gamma \square_{\Gamma} M \longrightarrow (\Gamma \otimes_A B) \square_{\Sigma} (B \otimes_A M) \]

where \( f(\gamma \otimes m) = \gamma \otimes 1 \otimes 1 \otimes m. \) The back adjunction \( \beta_N : \pi_* \pi^* N \to N \) is the \( B \)-linear extension of the top composite in the commutative diagram in which \( g(\gamma \otimes b \otimes n) = \pi(\gamma) \eta_b (b) \otimes n. \)
An ideal \( I \subseteq A \) is \emph{invariant} iff the ideal in \( \Gamma \) generated by \( \eta_h(I) \) coincides with the ideal generated by \( \eta_n(I) \). The most elementary change of rings theorem is now the following.

**Proposition 1.3.** Let \( I \subseteq A \) be an invariant ideal, \( \bar{A} = A/I, \bar{\Gamma} = \Gamma/\Gamma I \). Then:

(a) There is a unique Hopf algebroid structure on \( (\bar{A}, \bar{\Gamma}) \) such that the projection \( \pi: (A, \Gamma) \to (\bar{A}, \bar{\Gamma}) \) is a map of Hopf algebroids.

(b) For \( M \subseteq (\Gamma\text{-comod}) \), \( \beta_M \) is an isomorphism. For \( N \subseteq (\Gamma\text{-comod}) \), \( \alpha_N \) is the projection \( N \to N/IN \).

(c) The adjoint pair (1.2) gives an equivalence between \( (\Gamma\text{-comod}) \) and the full subcategory of \( (\Gamma\text{-comod}) \) generated by \( N \) such that \( IN = 0 \).

(d) If \( IN = 0 \) then

\[
\pi_*: \text{Ext}_\Gamma^* (A, N) \to \text{Ext}_\Gamma^* (\bar{A}, N)
\]

is an isomorphism.

\textbf{Proof.} (a) and (b) are immediate, and (b) implies (c). (d) follows from the fact that \( \pi_*: \Omega(\Gamma; N) \to \Omega(\bar{\Gamma}; N) \) is an isomorphism.

**Proposition 1.4.** ("Shapiro's Lemma"). Let \( \pi: (A, \Gamma) \to (B, \Sigma) \) be a morphism of Hopf algebroids. Let \( \Gamma \) be \( A \)-flat and assume that \( \Gamma \otimes_A B \) is injective as a right \( \Sigma \)-comodule. Then

\[
\text{Ext}_\Gamma^* (A, \pi_* N) \cong \text{Ext}_\Sigma^* (B, N)
\]

naturally in the \( \Sigma \)-comodule \( N \), in such a way that for a \( \Gamma \)-comodule \( M \) the following diagram commutes.

\[
\begin{array}{ccc}
\text{Ext}_\Gamma^* (A, M) & \xrightarrow{\pi_*} & \text{Ext}_\Sigma^* (B, \pi_* M) \\
\text{Ext}_\Gamma^* (A, \alpha_M) & \downarrow & \\
\end{array}
\]

\textbf{Proof.} Let \( X \) be a right \( B \)-module such that \( \Gamma \otimes_A B \) is a summand of \( X \otimes_B \Sigma \).

Let \( N \to I_* \) be a resolution of \( N \) over \( \Sigma \). Since \( I_* \) is \( B \)-split exact, \( (X \otimes_B \Sigma) \otimes_z I_* \cong X \otimes_B I_* \) is exact, and thus the summand \( \pi^* I_* \cong (\Gamma \otimes_A B) \otimes_z I_* \) is exact as well. Furthermore if \( I^* \) is a summand of \( \Sigma \otimes_B Y \) for the left \( B \)-module \( Y \) then

\[
(\Gamma \otimes_A B) \otimes_z (\Sigma \otimes_B Y) \cong (\Gamma \otimes_A B) \otimes_B Y \cong \Gamma \otimes_A Y
\]

is an extended \( \Gamma \)-comodule. Thus its summand \( \pi^* I^* = (\Gamma \otimes_A B) \otimes_z I^* \) is an injective \( \Gamma \)-comodule. The isomorphism now follows by adjointness and Lemma 1.1.
Commutativity of the diagram follows from commutativity of the corresponding diagram for Hom.

Section 2. The height $n$ change of rings theorem.

We now turn our attention to the Hopf algebroid $(BP_*, BP_*)$; thus a "comodule" will be a $BP_*/BP_*$-comodule. Recall [7], [5], that the only invariant prime ideals in $BP_*$ are $I_n = (p, v_1, \ldots, v_{n-1}), 0 \leq n \leq \infty$. For $0 \leq n < \infty$ let $BP_*^n$ denote the full subcategory of comodules $M$ which are finitely presented over $BP_*$ and killed by $I_n$. By Proposition 1.3 this is equivalent to the category of finitely presented $BP_*/BP_*/I_n (BP_*/BP_*)$-comodules.

The following generalizations of two theorems of P. S. Landweber will be useful. Their proofs are just as in [8], [9], once we observe that the invariant prime ideals of $BP_*/I_n$ pull back to the ideals $I_m, m \geq n$, in $BP_*$.

**Proposition 2.1.** $M \in BP_*^n$ has a finite filtration by subcomodules $F_i M$ such that for each $i, F_i M / F_{i-1} M$ is a suspension of $BP_*/I_k$ for some $k$ with $n \leq k < \infty$.

**Proposition 2.2.** Let $G$ be a $BP_*/I_n$-module. $G \otimes_{BP_*} -$ is exact on $BP_*^n$ iff $v_n$ is a nonzerodivisor on $G/I_n$ for all $k$ with $n \leq k < \infty$.

**Definition 2.3.** A comodule $M$ is of height $n$ iff $I_n M = 0$ and $v_n | M$ is an isomorphism.

If $M$ is a comodule killed by $I_n$ then $v_n | M$ is a comodule map (because $v_nv_n \equiv v_n$ (mod $I_n$)), so

$$v_n^{-1} M = \lim_{\rightarrow v_n} M$$

is a comodule, of height $n$. In particular, let $B(n)_* = v_n^{-1}BP_*/I_n$. Then for any comodule $M$ killed by $I_n$, $v_n^{-1} M = B(n)_* \otimes_{BP_*} M$.

Let $K(0)_* = \mathbb{Q}$ and $K(n)_* = \mathbb{F}[[v_n, v_n^{-1}]]$ for $n > 0$. There is an obvious ring-map $\pi : BP_* \rightarrow K(n)_*$. Using it, form

$$K(n)_* BP = K(n)_* \otimes_{BP_*} BP_* BP$$

$$BP_*K(n) = BP_* BP \otimes_{BP_*} K(n)$$

$$K(n)_*K(n) = K(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} K(n)_* .$$

Clearly $K(n)_*K(n)$ is a commutative Hopf algebra over $K(n)_*$, and there is a natural map of Hopf algebroids $\pi : BP_*BP \rightarrow K(n)_*K(n)$.

Since $I_n$ is invariant, we have a factorization

$$B(n)_* \overset{\eta_R}{\longrightarrow} K(n)_* \square_{K(n)_*K(n)} K(n)_*BP$$

$$\downarrow$$

$$BP_* \longrightarrow BP_* BP \longrightarrow K(n)_*BP$$

$\eta_R$
of ring-maps. Thus $K(n)_*BP$ is a right $B(n)_*$-module. It is also obviously a left $K(n)_*K(n)$-comodule, and we have:

**Proposition 2.4.** There is a map

$$K(n)_*BP \to K(n)_*K(n) \otimes_{K(n)_*} B(n)_*$$

which is an isomorphism of $K(n)_*K(n)$-comodules and of $B(n)_*$-modules, and which carries 1 to 1.

**Proof.** Our proof is a counting argument; and in order to meet requirements of connectivity and finiteness, we pass to suitable "valuation rings." Thus let

$$k(0)_* = Z_{(p)} \subset K(0)_*$$

$$k(n)_* = \mathbb{F}_p[v_n] \subset K(n)_*, \quad n > 0$$

$$k(n)_*BP = k(n)_* \otimes_{B(n)} BP_*BP \subset K(n)_*BP$$

$$b(n)_* = k(n)_*[u_1, u_2, \ldots] \subset B(n)_*$$

where $u_k = v_n^{-1}v_{n+k}$.

It follows from Theorem 1 of [15] that in $k(n)_*BP$,

$$\eta_R(v_{n+k}) = v_n t_n^{\alpha n} - v_n^{2^{n+1}t_n} \mod (\eta_R(v_{n+1}), \ldots \eta_R(v_{n+k-1})). \quad (2.5)$$

Hence $\eta_R : BP_* \to k(n)_*BP$ factors through an algebra map $b(n)_* \to k(n)_*BP$. It is clear from (2.5) that as a right $b(n)_*$-module, $k(n)_*BP$ is free on generators $t^n = t_1^n t_2^n \cdots$ where $0 \leq \alpha_i < p^n$ and all but finitely many $\alpha_i$ are 0; in particular it is of finite type over $b(n)_*$.

Now define

$$s(n)_* = k(n)_*BP \otimes_{b(n)_*} k(n)_* \subset K(n)_*K(n);$$

by the above remarks $s(n)_* = k(n)_*[t_1, t_2, \ldots]/(t_k^{p^n} - v_n^{-p^{n-1}t_k} : k \geq 1)$ as an algebra. $(k(n)_*, s(n)_*)$ is clearly a sub Hopf algebroid of $(K(n)_*, K(n)_*K(n))$; so $s(n)_*$ is a Hopf algebra over the principal ideal domain $k(n)_*$.

The natural map $BP_*BP \to s(n)_*$ makes $BP_*BP$ a left $s(n)_*$-comodule, and this induces a left $s(n)_*$-comodule structure on $k(n)_*BP$. We will show that the latter is an extended left $s(n)_*$-comodule.

Define a $b(n)_*$-linear map $f : k(n)_*BP \to b(n)_*$ by

$$f(t^n) = \begin{cases} 1 & \text{if } \alpha = (0, 0, \ldots) \\ 0 & \text{otherwise.} \end{cases}$$

Then $f$ satisfies the equations

$$f \eta_R = \text{id.} : b(n)_* \to b(n)_*$$

$$f \otimes_{b(n)_*} k(n)_* = \epsilon : s(n)_* \to k(n)_*.$$
\[
\begin{align*}
\begin{array}{c}
k(n)_* \otimes_{b(n)_*} k(n)_* \otimes_{b(n)_*} k(n)_* \rightarrow s(n)_* \otimes_{k(n)_*} k(n)_* \otimes_{k(n)_*} s(n)_* \\
\end{array}
\end{align*}
\]

Since \(\psi \eta_R(x) = 1 \otimes \eta_R(x)\), \(\psi\) is \(b(n)_*\)-linear, so \(\tilde{j}\) is too. We claim \(\tilde{j}\) is an isomorphism. Since both sides are free of finite type over \(b(n)_*\) it suffices to prove that \(\tilde{j} \otimes_{b(n)_*} k(n)_*\) is an isomorphism. But (2.6) is then reduced to

\[
\begin{align*}
\begin{array}{c}
s(n)_* \rightarrow s(n)_* \otimes_{b(n)_*} s(n)_* \\
\tilde{j} \otimes_{b(n)_*} k(n)_* \\
s(n)_* \otimes_{b(n)_*} s(n)_* \rightarrow s(n)_* \otimes_{b(n)_*} s(n)_* \\
\end{array}
\end{align*}
\]

so the claim follows from unitarity of \(\Delta\).

Now the map \(K(n)_* \otimes_{h(n)_*} \tilde{j}\) satisfies the requirements of the proposition.

**Corollary 2.7.** \(B(n)_* \rightarrow K(n)_* \Box_{K(n)_*} K(n)_* \otimes_{K(n)_*} B(n)_*\) is an isomorphism of \(B(n)_*\)-modules.

**Proof.** The natural isomorphism

\[
B(n)_* \rightarrow K(n)_* \Box_{K(n)_*} K(n)_* \otimes_{K(n)_*} B(n)_*
\]

is \(B(n)_*\)-linear and carries 1 to 1. Hence

\[
\begin{array}{c}
\approx \\
\begin{array}{c}
B(n)_* \\
\eta_R \\
K(n)_* \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\Box_{K(n)_*} K(n)_* \\
\otimes_{K(n)_*} B(n)_*
\end{array}
\]

commutes, and \(\eta_R\) is an isomorphism.

For any comodule \(M\) we have a natural factorization

\[
\begin{array}{c}
M \rightarrow \pi^* \pi_* M \\
\alpha_M
\end{array}
\]

of the adjunction morphism \(\alpha_M\).

**Proposition 2.8.** \(\alpha_M\) is an isomorphism.

**Proof.** We shall prove below (Lemma 2.11) that every comodule is a direct limit of finitely presented comodules. Thus we may assume that \(M\) is finitely presented. Replacing \(M\) by \(M/IM\) leaves source and target unaltered, so we may assume that \(M \in \mathcal{B}E_\ast\).

Proposition 2.2 implies that \(B(n)_* \otimes_{B_\ast} K(n)_* \otimes_{B_\ast} \) are exact on \(\mathcal{B}E_\ast\); and by Proposition 2.4, \(B_\ast \otimes_{K(n)_*} K(n)_*\) is exact on \((K(n)_* K(n)_*)\)-comod).
Thus if $M \in BP_*$ is filtered as in Proposition 2.1, then both source and target of $\alpha_M$ are filtered, and by induction using the 5-lemma we are reduced to considering $M = BP_*/I_k$, $k \geq n$.

If $M = BP_*/I_k$ with $k > n$, then both source and target are 0. If $M = BP_*/I_n$, it is easy to see that $\alpha_M$ is transformed by the conjugation $c$ to

$$\tilde{\eta}_B : B(n)_* \to K(n)_* \otimes_{K(n)_*} K(n)_* BP,$$

which is an isomorphism by Corollary 2.6.

**Theorem 2.9.** The category of $BP_*$-BP-comodules of height $n$ is naturally equivalent to the category of $K(n)_* K(n)$-comodules.

**Proof.** The functors giving the equivalence are of course $\pi^* = BP_* K(n) \otimes_{K(n)_* K(n)}$ and $\pi_* = K(n)_* \otimes_{BP_*}$. Now $\beta_N : \pi_* \pi^* N \to N$ is clearly an isomorphism for any $K(n)_* K(n)$-comodule $N$. On the other hand we identified $\alpha_M$ with the map $M \to B(n)_* \otimes_{BP_*} M$, which is an isomorphism exactly when $M$ is of height $n$.

**Theorem 2.10.** The natural map

$$\text{Ext}_{BP_*BP_*} (BP_*, M) \to \text{Ext}_{K(n)_*K(n)_*} (K(n)_*, K(n)_* \otimes_{BP_*} M)$$

is an isomorphism if $M$ is of height $n$.

**Proof.** By Proposition 2.4, our Shapiro’s Lemma applies, and the result follows from Proposition 2.8.

We now turn to a proof of

**Lemma 2.11.** Every $BP_*$-BP-comodule is a direct limit of finitely presented comodules.

We begin with a result due to P. S. Landweber ([9], Prop. 2.4).

**Lemma 2.12.** Every element of a $BP_*$-BP-comodule $M$ is in the image of a comodule map from a comodule which is free and finitely generated over $BP_*$.

**Proof.** First let $N$ be a right $BP_*$-BP-comodule which is free and finitely generated over $BP_*$, and write $\psi(x) = \sum_a r_a(x) \otimes t^a \in N \otimes_{BP_*} BP$. $N^* = \text{Hom}_{BP_*} (N, Z)$ is naturally a left $BP_*$-module, and in fact a left $BP_*$-comodule with $\psi(y) = \sum_z z^a \otimes r_a(y)$, where $z_i$ is the conjugate $c_t$ of $t_i$, and where $r_a^* = \text{Hom}_{Z_{(p)}} (r_a, Z_{(p)})$.

In particular, for $d \geq 0$ let $N(d)$ be the right sub $BP_*$-module of $BP_*$ generated by $\{z^a : |a| \leq d\}$, where $|a| = 2\Sigma a_i (p^i - 1)$ is the dimension of $z^a$. Since $\Delta(z^a)$ is homogeneous, this is a subcomodule of $BP_*$.

Now let $M$ be an arbitrary comodule, and let $x \in M$. There exists an integer $d$ such that $\psi(x) \in N(d) \otimes_{BP_*} M$. By associativity, $\psi(x)$ corresponds under the isomorphism $N(d) \otimes_{BP_*} M \simeq \text{Hom}_{BP_*} (N(d)^*, M)$ to a comodule map. If $y \in N(d)^*$ is such that
\[ \langle x^\alpha, y \rangle = \begin{cases} 1 & \alpha = 0 \\ 0 & \alpha \neq 0 \end{cases} \]

then \( y \) hits \( x \), and we are done.

Note that \( N(d)^* \) is the \( BP \)-homology of the Spanier-Whitehead dual of the \( d \)-skeleton of \( BP \).

**Corollary 2.13.** Every \( BP_*BP \)-comodule is the direct limit of its finitely generated sub-comodules.

**Corollary 2.14.** Every \( BP_*BP \)-comodule \( M \) is a quotient of a comodule \( F \) which is \( BP_* \)-free. If \( M \) is finitely generated over \( BP_* \) then we may take \( F \) to be finitely generated over \( BP_* \).

**Proof of Lemma 2.11.** By Corollary 2.13 we may assume that \( M \) is finitely generated. By Corollary 2.14 there is a short exact sequence

\[ 0 \to R \to F \to M \to 0 \]

of comodules such that \( F \) is free and finitely generated over \( BP_* \). Let \( \Sigma \) be the directed set of finitely generated subcomodules \( S \) of \( R \). Then by Corollary 2.13 and the exactness of \( \lim \),

\[ M = \lim_{s \in \Sigma} F/S; \]

and each \( F/S \) is finitely presented.

**Section 3. The \( v_n \)-local change of rings theorem.**

In this section we will generalize the change of rings theorem of Section 2 to a larger category of comodules. Instead of requiring that \( I_* \) annihilate the comodule, we demand only that each element be killed by some power of \( I_* \).

**Definition 3.1.** Let \( A \) be a commutative ring and \( I \subset A \) an ideal. An \( A \)-module \( M \) is \( I \)-nil iff for each \( x \in M \) there is an integer \( k \) such that \( I^kx = 0 \). \( M \) is \( I \)-nilpotent iff for some \( k \), \( I^kM = 0 \).

These notions coincide when \( M \) is finitely generated.

This condition is related to the invertibility of \( v_n \) by the following Lemma.

**Lemma 3.2.** If the \( BP_*BP \)-comodule \( M \) is \( I_- \)-nil, then there exists a unique \( BP_*BP \)-comodule structure on

\[ v_n^{-1}M = \lim_{/\phantom{v_n}} M \]

such that the localization map \( M \to v_n^{-1}M \) is a map of comodules.

**Proof.** Let \( M' \subset M \) be a finitely generated submodule. Then \( M' \) is \( I_- \)-nilpotent; say \( I_+^kM' = 0 \). Then ([11] : 3.6) multiplication by \( v_n^{k-1} \) on \( M' \)
is a comodule map. Hence $v_n^{-1}M'$, regarded as the direct limit of the system

\[
\begin{array}{ccc}
v_n^k & \rightarrow & M' \\
& v_n^{k+1} & \rightarrow \end{array}
\]

is a comodule in such a way that $M' \rightarrow v_n^{-1}M'$ is a comodule map. Also it is clear that if $M' \subset M'' \subset M$ with $M''$ finitely generated then $v_n^{-1}M' \rightarrow v_n^{-1}M''$ is a comodule map. Thus $v_n^{-1}M = \lim_m v_n^{-1}M'$, the limit taken over finitely generated subcomodules of $M$, is a comodule. Uniqueness is clear.

**Definition 3.3.** A $BP_\ast BP$-comodule $M$ is $v_\ast$-local iff $v_\ast$ acts bijectively on $M$.

**Example 3.4.** Let $N_\ast^0 = BP_\ast / I_\ast$. Define comodules $N_\ast$, $M_\ast$ inductively as follows. Suppose that $N_\ast^i$ has been defined and is $I_\ast^{i+1}$-nil. Then $M_\ast^i = v_{\ast+1}^{-1}N_\ast^i$ is a comodule by Lemma 3.2, and the exact sequence

$$0 \rightarrow N_\ast^i \rightarrow M_\ast^i \rightarrow N_\ast^{i+1} \rightarrow 0$$

defines an $I_{\ast+1}$-nil comodule $N_\ast^{i+1}$. $M_\ast^i$ is $v_{\ast+1}$-local. This example plays a central role in [13].

We postpone to the end of this section a proof of the following result of Peter Landweber. We are grateful to him for allowing us to include it.

**Proposition 3.5.** Any $v_\ast$-local comodule is $I_\ast$-nil.

We next describe the replacement in this context for $K(n)_\ast$.

**Definition 3.6.** Let $E(\ast)^\ast = \mathbb{Z}([v_1, \cdots, v_n, v_n^{-1}])$ with the $BP_\ast$-algebra structure sending $v_i$ to 0 for $i > n$.

**Remark 3.7.** Let $E_\ast$ be any $BP_\ast$-module on which the sequence $p, v_1, \cdots$ is regular: that is, such that $v_\ast$ acts injectively on $E_\ast / I_\ast E_\ast$ for all $n \geq 0$. Then by Prop. 2.2 $E_\ast \otimes_{BP_\ast}$ is exact on $BP_\ast$, so by Lemma 2.11 $E_\ast \otimes_{BP_\ast}$ is exact on the category of $BP_\ast$-comodules.

Now suppose $E_\ast$ is a commutative $BP_\ast$-algebra on which $p, v_1, \cdots$ is regular. Then define

$$E_\ast E = E_\ast \otimes_{BP_\ast} BP_\ast BP \otimes_{BP_\ast} E_\ast.$$

$E_\ast E$ is a Hopf algebroid by extension of the structure maps for $BP_\ast BP$. We claim that $E_\ast E$ is $E_\ast$-flat. For an $E_\ast$-module $M$,

$$E_\ast E \otimes_{E_\ast} M = E_\ast \otimes_{BP_\ast}(BP_\ast BP \otimes_{BP_\ast} M).$$

By the above remarks and the flatness of $BP_\ast BP$, this is an exact functor of $M$, as desired.

**Topological Remark 3.8.** If $E_\ast$ is a $BP_\ast$-module on which $p, v_1, \cdots$, acts regularly, then by the above remarks $E_\ast(X) = E_\ast \otimes_{BP_\ast} BP_\ast(X)$ defines an
additive homology theory on spectra. Thus there is a spectrum $E$ such that $E_*(X) \cong \pi_*(E \wedge X)$ naturally in $X$; see [2], Remark 6.5. Clearly

$$\pi_*(E \wedge E) \cong E_* \otimes_{BP_*} BP_* \otimes_{BP_*} E_*.$$  \hspace{1cm} (3.9)

In any case if $E_* = E(n)_*$, one can show from the work of [18] and [19] that $E(n)$ is an associative ring-spectrum in such a way that the map $BP_*(\ ) \rightarrow E(n)_*(\ )$ is multiplicative. By Remark 3.7, the right-hand side of (3.9) is $E(n)_*$-flat, so $\pi_*(E \wedge E)$ is the usual [1] Hopf algebroid of cooperations and (3.9) is an isomorphism of Hopf algebroids. Thus there is an Adams spectral sequence with

$$E_2^* = \Ext_{E(n)_*E(n)_*}(E(n)_*, E(n)_*(X)),$$

natural in the spectrum $X$.

For example, $E(0)_*(\ )$ is rational homology theory, and $E(1)_*(\ )$ is a factor of complex K-theory localized at $p$.

The main theorem of this section is

**Theorem 3.10.** If $M$ is a $v_\gamma$-local comodule then the natural map

$$\Ext_{BP_*BP_*}(BP_*, M) \rightarrow \Ext_{E(n)_*E(n)_*}(E(n)_*, E(n)_* \otimes_{BP_*} M)$$

is an isomorphism.

The proof will use a couple of lemmas.

**Lemma 3.11.** Any $v_\gamma$-local comodule is a direct limit of comodules $v^{-1}M$ where $M$ is finitely presented and $I_\gamma$-nilpotent.

**Proof.** Write $M = \lim_i M_i$ as a direct limit of finitely presented comodules by Lemma 2.11. $M_i$ is finitely generated and $I_\gamma$-nilpotent (since $M$ is $I_\gamma$-nil); let $k(\sigma)$ be the least integer $k$ such that $I^kM_i = 0$. Then $i_\gamma$ factors through $M_i/I^k\gamma(M_i)$. Note that if $\sigma \leq \tau$ in $\Sigma$ then $k(\sigma) \leq k(\tau)$, so the comodules $M_i/I^k\gamma(M_i)$ form a directed system with limit $M$. Similarly, $M_\gamma/I^k\gamma(M_\gamma)$ is $I_\gamma$-nil so $v^{-1}M_\gamma/I^k\gamma(M_\gamma)$ is a comodule. Now the system of these comodules has direct limit $M = v^{-1}M$ since localization commutes with direct limits.

**Lemma 3.12.** If $M$ is a finitely presented $I_\gamma$-nilpotent comodule then $v^{-1}M$ has a finite filtration with quotients isomorphic to suspensions of $B(n)_\gamma$.

**Proof.** Consider a Landweber filtration of $M$. Since any quotient of an $I_\gamma$-nilpotent module is $I_\gamma$-nilpotent, the associated quotients are suspensions of $BP_*/I_k$ for $k \geq n$. Now invert $v_\gamma$. Since localization is exact we obtain a filtration with quotients

$$v^{-1}BP_*/I_k = \begin{cases} B(n)_* & \text{if } k = n \\ 0 & \text{if } k > n. \end{cases}$$
Proof of Theorem 3.10. We may assume $M = v_n^{-1}M'$ with $M'$ finitely presented and $I_n$-nilpotent, as the general case then follows from Lemma 3.11. By Lemma 3.12 and Prop. 2.1 it suffices to prove the theorem for $M' = BP_*/I_n$. So consider the commutative diagram

$$
\begin{array}{c}
\pi_1 \downarrow \\
\text{Ext}^*_E(BP_*, B(n)_*) \xrightarrow{\pi_2} \text{Ext}^*_K(K(n)_*, K(n)_n)
\end{array}
$$

$\pi_2$ is an isomorphism by Prop. 1.3, and $\pi_3$ is an isomorphism by Theorem 2.10, so $\pi_1$ is an isomorphism.

We turn now to a proof of Proposition 3.5.

Lemma 3.13. Let $n > k \geq 0$ and let $M$ be a $BP_*$-$BP_*$-comodule such that $v_nM = M$ and $I_kM = 0$. Then each element of $M$ is $v_k$-torsion.

Proof. Suppose first that $x \in M$ is primitive—i.e. $\psi(x) = 1 \otimes x$—and let $x = v_\alpha y$. Recall ([5] Lemma 1.7) that modulo $I_k$, $v_n = v_kt_{n-\epsilon}^x + \cdots$, where the other terms involve $t^\tau$ with $|\alpha| < |\beta|\Delta_{n-k}$; here $\Delta_{n-k}$ is the multiindex with 1 in the $(n-k)$th place and 0 elsewhere. Now let $\beta_0$ be a multiindex of maximal dimension such that $v_\beta(y) = 0$. Then the coefficient of $z^{\alpha_\beta + \beta_\alpha} = v_\beta(y)$ in

$$
1 \otimes x = \psi(v_\alpha y) = \sum_{\alpha, \beta} z^{\alpha_\beta + \beta_\alpha} \otimes v_\alpha(y)
$$

is $v_\beta(y)$, which is thus 0.

Next let $\beta_1$ be a multiindex of maximal dimension such that $v_\beta(y) \neq 0$; thus $|\beta_1| < |\beta_0|$. Multiply (3.14) by $v_\alpha$ and observe that the coefficient of $z^{\alpha_\beta + \beta_\alpha} = v_\beta(y)$, which is thus 0.

Continuing, we find that $r_\beta(y)$ is $v_k$-torsion for all $\beta$, and in particular for $\beta = 0$. That is, $y$ is $v_k$-torsion; so $x = v_\alpha y$ is too, q.e.d.

Now let $x$ be arbitrary. Let $\beta_0$ be a multiindex of maximal dimension such that $r_\beta(y) \neq 0$. Then for $\gamma \neq 0$, $r_\gamma v_\beta(x) = \sum_\alpha a_{\gamma, \beta} v_\alpha(x) = 0$—i.e., $r_\beta(x)$ is primitive, and hence $v_k$-torsion. Proceeding by induction, we find that $x$ is primitive mod $v_k$-torsion, and hence is $v_k$-torsion.

Proof of Proposition 3.5. Since $I_0 = 0$, $I_nM = 0$, and Lemma 3.13 implies that $M$ is $I_1$-nil. Suppose inductively that $M$ is $I_k$-nil for some $k < n$. Let $F_1M = \{x \in M : I_kx = 0\}$; it is a submodule since $I_k$ is invariant. Multiplication by $v_\alpha$ on $F_1M$ is clearly monic, and we claim that it is epic as well. Let $x \in F_1M$; there exists $y \in M$ such that $x = v_\alpha y$. Then $0 = I_kx = v_\alpha y$ implies $I_ky = 0$ since $v_\alpha M = 0$. Now define a filtration of $M$ inductively by means of the pull-back diagram

$$
\begin{array}{c}
0 \\ \\
F_1M \\ \\
M \\
M/F_1M
\end{array}
$$

where

$$
0 \rightarrow F_1M \rightarrow F_{i+1}M \rightarrow F_i(M/F_1M) \rightarrow 0
$$

and

$$
0 \rightarrow F_1M \rightarrow M \rightarrow M/F_1M \rightarrow 0.
$$
By induction $F_{i+1}M/F_iM$ is $v_n$-local and killed by $I_k$, and hence by Lemma 3.13 is $v_k$-torsion. So $F_{i+1}M$ is $I_{k+1}$-nil for all $i$. Since $M$ is $I_k$-nil, $M = \lim_{\to} F_iM$ and hence $M$ is $I_{k+1}$-nil. The proposition now follows by induction on $k$.


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