A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

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The slice spectral sequence

An equivariant approach to the Riemann Hypothesis

Unni Namboodiri Lectures
University of Chicago

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The object is to show that all nontrivial zeros have first coordinate on the critical line. The group $C_2$ acts by complex conjugation. Using the functional equation we can modify the zeta function to get a new function $\Lambda$ that is symmetric about the critical line. This leads to an action of $G = C_2 \times C_2$ on $CP^\infty$ for which modified zeta function is equivariant.
An equivariant approach to the Riemann Hypothesis (continued)

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Hence the problem is very similar to the Kervaire invariant question except that the group involved is not cyclic. The Slice Theorem (to be explained below) still holds, but the slices themselves are more complicated because of the bigger group. Using the techniques we have developed in the cyclic case, there is a good chance we can do the necessary calculations here and arrive at a similar proof.
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A solution to the Arf-Kervaire invariant problem I: History and background

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Vic Snaith and Bill Browder in 1981
Photo by Clarence Wilkerson
A wildly popular dance craze

Can you do the Arf Invariant?

Is it a jig or a reel?

Drawing by Carolyn Snaith 1981
London, Ontario
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Mike Hill, myself and Mike Hopkins
Photo taken by Bill Browder
February 11, 2010
Our main result

Our main theorem can be stated in three different but equivalent ways:
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- **Manifold formulation**: It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds $M$ is always zero.

- **Stable homotopy theoretic formulation**: It says that certain long sought hypothetical maps between high dimensional spheres do not exist.

- **Unstable homotopy theoretic formulation**: It says something about the EHP sequence, which has to do with unstable homotopy groups of spheres.
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The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved.
Snaith’s book

**Stable Homotopy Around the Arf-Kervaire Invariant**, published in early 2009,
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*Stable Homotopy Around the Arf-Kervaire Invariant*, published in early 2009, just before we proved our theorem.

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Snaith’s book (continued)

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Our main result (continued)

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**Main Theorem**

*The Arf-Kervaire elements* \( \theta_j \in \pi_{2j+1-2+n}(S^n) \) *for large* \( n \) *do not exist for* \( j \geq 7 \).*
Here is the stable homotopy theoretic formulation.

**Main Theorem**

*The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1} - 2 + n}(S^n)$ for large $n$ do not exist for $j \geq 7$.*

The $\theta_j$ in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial.
Here is the stable homotopy theoretic formulation.

**Main Theorem**

\[ \text{The Arf-Kervaire elements } \theta_j \in \pi_{2j+1-2+n}(S^n) \text{ for large } n \text{ do not exist for } j \geq 7. \]

The \( \theta_j \) in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It follows from Browder's theorem of 1969 that such things can exist only in dimensions that are 2 less than a power of 2.
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After 1980, the problem faded into the background because it was thought to be too hard. Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then.
Pontryagin’s early work on homotopy groups of spheres

Lev Pontryagin 1908-1988

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- Pick a regular value \( y \in S^n \). Its inverse image will be a smooth \( k \)-manifold \( M \) in \( S^{n+k} \).
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.
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Pontryagin’s early work (continued)

Let $D^n$ be the closure of an open ball around a regular value $y \in S^n$. If it is sufficiently small, then $V^{n+k} = f^{-1}(D^n) \subset S^{n+k}$ is an $(n+k)$-manifold homeomorphic to $M \times D^n$ with boundary homeomorphic to $M \times S^{n-1}$. 
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There is a way to reverse this procedure. A framed manifold $M^k \subset S^{n+k}$ determines a map $f : S^{n+k} \to S^n$. 
Pontryagin’s early work (continued)

To proceed further, we need to be more precise about what we mean by continuous deformation.

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Pontryagin’s early work (continued)

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Two maps $f_1, f_2 : S^{n+k} \to S^n$ are **homotopic** if there is a continuous map $h : S^{n+k} \times [0, 1] \to S^n$ (called a **homotopy** between $f_1$ and $f_2$) such that
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h(x, 0) = f_1(x) \quad \text{and} \quad h(x, 1) = f_2(x).
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If $y \in S^n$ is a regular value of $h$, then $h^{-1}(y)$ is a framed $(k + 1)$-manifold $N \subset S^{n+k} \times [0, 1]$ whose boundary is the disjoint union of $M_1 = f_1^{-1}(y)$ and $M_2 = f_2^{-1}(y)$. This $N$ is called a \textbf{framed cobordism} between $M_1$ and $M_2$. When it exists the two closed manifolds are said to be \textbf{framed cobordant}. 
Pontryagin’s early work (continued)
Here is an example of a framed cobordism for $n = k = 1$. 

\[ \begin{array}{c}
\text{Pontryagin (1930's)} \\
M_1 \\
M_2 \\
N \\
\end{array} \]

\begin{align*}
\text{Framed cobordism}
\end{align*}
Pontryagin’s early work (continued)

\[ \Omega_k := \{ \text{stably framed } k\text{-manifolds} \}/\text{cobordism} \]

**Theorem:** The above construction gives a bijection

\[ \pi_{n+k}(S^n) \approx \Omega_k \]

where

\[ \pi_{n+k}(S^n) := \{ \text{maps } S^{n+k} \to S^n \}/\text{homotopy} \]
Pontryagin’s early work (continued)

Pontryagin (1930’s)

\[ k=0 \]
Pontryagin’s early work (continued)

Pontryagin (1930’s)

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Pontryagin’s early work (continued)

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\[ k=0 \]
Pontryagin’s early work (continued)

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\[ \pi_n(S^n) = \mathbb{Z} \]

k=0
Pontryagin’s early work (continued)

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(the degree)

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\(\pi_n(S^n) = \mathbb{Z}\) (the degree)

\(k=0\)

\(k=1\)
Pontryagin’s early work (continued)

Pontryagin (1930’s)

\[ \pi_n(S^n) = \mathbb{Z} \]

(the degree)

\[ \pi_{n+1}(S^n) = \mathbb{Z}/2 \]
Pontryagin’s early work (continued)

Pontryagin (1930’s)

\[ k = 2 \]
Pontryagin’s early work (continued)

Pontryagin (1930’s)

\[ k=2 \quad \text{genus } M = 0 \implies M \text{ is a boundary} \]

(since \( S^2 \) bounds a disk and \( \pi_2(\text{GL}_n(\mathbb{R}))=0 \))
Pontryagin’s early work (continued)

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Suppose the genus of \( M \) is greater than 0.
Pontryagin’s early work (continued)

Pontryagin (1930’s)

k=2
Pontryagin’s early work (continued)
Pontryagin’s early work (continued)

Pontryagin (1930’s)

k=2

choose an embedded arc
Pontryagin’s early work (continued)

Pontryagin (1930’s)

1.34

Pontryagin (1930’s)

k=2

choose an embedded arc

cut the surface open and glue in disks
Pontryagin’s early work (continued)

Pontryagin (1930’s)

\( k=2 \)

framed surgery
Pontryagin’s early work (continued)

Pontryagin (1930’s)

Obstruction: \( \varphi : H_1(M; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \)
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Conclusion: \( \Omega_2 = \pi_{n+2}(S^n) = 0. \)
Pontryagin’s mistake for $k = 2$
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The Arf invariant of a quadratic form in characteristic 2

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In other words, $\overline{H}$ has a basis for which the bilinear form’s matrix has the symplectic form

$$
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 1 \\
1 & 0 \\
1 & 0 \\
\ddots & \ddots \\
0 & 1 \\
1 & 0
\end{bmatrix}.
$$
The Arf invariant of a quadratic form in characteristic 2
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In 1941 Arf proved that this invariant (along with the number $n$) determines the isomorphism type of $q$. 
Money talks: Arf’s definition republished in 2009

Cahit Arf 1910-1997
The Kervaire invariant of a framed $(4m + 2)$-manifold

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For \(m = 0\), Kervaire’s \(q\) coincides with Pontryagin’s \(\varphi\).
The Kervaire invariant of a framed \((4m + 2)\)-manifold (continued)

What can we say about \(\Phi(M)\)?
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![Topology circa 1960: Kervaire’s example](image)

\(X = N/\partial N\)

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The Kervaire invariant of a framed \((4m + 2)\)-manifold (continued)

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\[ \text{Brown-Peterson (1966) showed that it vanishes for all positive even } m. \]

Ed Brown 1930-2000

Frank Peterson
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A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

Background and history

Our main result
Pontryagin’s early work

The Arf-Kervaire formulation

Questions raised by our theorem

Our strategy
Ingredients of the proof
The spectrum \( \Omega \)
How we construct \( \Omega \)

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Questions raised by our theorem

Adams spectral sequence formulation.
We now know that the $h_2$ for $j \geq 7$ are not permanent cycles, so they have to support nontrivial differentials.
We have no idea what their targets are.

Unstable homotopy theoretic formulation.
In 1967 Mahowald published an elaborate conjecture about the role of the $\theta_j$ (assuming that they all exist) in the unstable homotopy groups of spheres.
Since they do not exist, a substitute for his conjecture is needed.
We have no idea what it should be.

Our method of proof offers a new tool, the slice spectral sequence, for studying the stable homotopy groups of spheres.
We look forward to learning more with it in the future.
We will illustrate it at the end of the talk.
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This means

\[ \Sigma X \cong Y \]

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Fiber sequences and cofiber sequences are the same, up to weak equivalence.

While space \( X \) has a homotopy group \( \pi_k(X) \) for each positive integer \( k \), a spectrum \( X \) has an abelian homotopy group \( \pi_k(X) \) defined for every integer \( k \).

For the sphere spectrum \( S_0 \), \( \pi_k(S_0) \) is the usual homotopy group \( \pi_{n+k}(S_n) \) for \( n > k + 1 \).

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John Milnor

Sergei Novikov

Dan Quillen
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- We also make use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group $G$ (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers $\mathbb{Z}$, but by $RO(G)$, the real representation ring of $G$. 
Ingredients of the proof (continued)

More ingredients of our proof:

- We also make use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group $G$ (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers $\mathbb{Z}$, but by $RO(G)$, the real representation ring of $G$. Our calculations make use of this richer structure.
The spectrum $\Omega$

We will produce a map $S^0 \to \Omega$, where $\Omega$ is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.
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If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for $\theta_j$ for larger $j$ is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \text{ mod } 256$ for $j \geq 7$. 
How we construct $\Omega$

Our spectrum $\Omega$ will be the fixed point spectrum for the action of $C_8$ (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.

To construct it we start with the complex cobordism spectrum $\text{MU}$. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of $C_2$ defined by complex conjugation. The fixed point set of this action is the set of real points, known to topologists as $\text{MO}$, the unoriented cobordism spectrum.

In this notation, $U$ and $O$ stand for the unitary and orthogonal groups.
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Some people who have studied $MU$ as a $C_2$-spectrum:

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- Shoro Araki (1930–2005)
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This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.
A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

Background and history
Our main result
Pontryagin's early work
The Arf-Kervaire formulation
Questions raised by our theorem

Our strategy
Ingredients of the proof
The spectrum $\Omega$
How we construct $\Omega$

The slice spectral sequence

A homotopy fixed point spectral sequence
The corresponding slice spectral sequence