A solution to the Arf-Kervaire invariant problem III: The Gap Theorem

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The main theorem

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To prove this we produce a map $S^0 \to \Omega$, where $\Omega$ is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each $\theta_j$ is nontrivial. This means that if $\theta_j$ exists, we will see its image in $\pi_*(\Omega)$.

(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of $k$ modulo 256.

(iii) Gap Theorem. $\pi_{-2}(\Omega) = 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence and is the subject of this talk.
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How we construct $\Omega$

Our spectrum $\Omega$ will be the fixed point spectrum for the action of $C_8$ (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$. 
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- $MU$ has an action of the group $C_2$ via complex conjugation.
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- $H_\ast(MU; \mathbb{Z}) = \mathbb{Z}[b_i : i > 0]$ where $|b_i| = 2i$.
- $\pi_\ast(MU) = \mathbb{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring.
How we construct \( \Omega \) (continued)

Given a spectrum \( X \) acted on by a group \( H \) of order \( h \) and a group \( G \) of order \( g \) containing \( H \), there are two formal ways to construct a \( G \)-spectrum from \( X \):

(i) The transfer. The spectrum \( \hat{Y} = \pi^+ \wedge H X \) underlain by \( \bigsqcup_{g/h} X \) has an action of \( G \) which permutes the wedge summands, each of which is invariant under \( H \). This is used to construct our slice cells \( \hat{S}^{(m \rho H)} = \pi^+ \wedge H S_{m \rho H} \).

(ii) The norm. The spectrum \( \mathcal{N}_G H X \) underlain by \( \bigsqcup_{g/h} X \) has an action of \( G \) which permutes the smash factors, each of which is invariant under \( H \). This was described in the last lecture.
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$$MU_R^{(4)} = N^G_H MU_R.$$

It has homotopy groups $\pi_*^G MU_R^{(4)}$ indexed by the representation ring $RO(G)$. 

Let $\rho_G$ denote the regular representation of $G$. We form a $G$-spectrum $\tilde{\Omega}$ by inverting a certain element $D \in \pi_{19}^G MU_R^{(4)}$. Our spectrum $\Omega$ is its fixed point set, $\Omega = \tilde{\Omega}^G$. 

$\textbf{MU}$
- Basic properties
- Refining homotopy
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The slice filtration on $\mathcal{N}_H^G \mathbb{M}U_R$

We want to study

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where $H = C_2$ and $G = C_{2n+1}$. 
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**Definition**

Suppose $X$ is a $G$-spectrum such that its underlying homotopy group $\pi^u_k(X)$ is free abelian.
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It has a slice filtration and we need to identify the slices. The following notion is helpful.

**Definition**

Suppose $X$ is a $G$-spectrum such that its underlying homotopy group $\pi^u_k(X)$ is free abelian. A refinement of $\pi^u_k(X)$ is an equivariant map

$$c : \hat{W} \to X$$

in which $\hat{W}$ is a wedge of slice cells of dimension $k$ whose underlying spheres represent a basis of $\pi^u_k(X)$. 

The refinement of $\pi^u_*(MU^{(4)}_{\mathbb{R}})$

Recall that $\pi_*(MU) = \pi^u_*(MU_{\mathbb{R}})$ is concentrated in even dimensions and is free abelian.
The refinement of $\pi_\ast^u(MU_R^{(4)})$

Recall that $\pi_\ast(MU) = \pi_\ast^u(MU_R)$ is concentrated in even dimensions and is free abelian. $\pi_{2k}^u(MU_R)$ is refined by an map from a wedge of copies of $\hat{S}(k\rho_2)$. 

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$\pi^u_*(MU^{(4)}_R)$ is a polynomial algebra with 4 generators in every positive even dimension.
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$$(-1)^j \quad r_i(1) \rightarrow r_i(2) \rightarrow r_i(3) \rightarrow r_i(4)$$
The refinement of $\pi_*^U(MU_R^{(4)})$ (continued)

\[ r_i(1) \xrightarrow{(-1)^i} r_i(2) \xrightarrow{} r_i(3) \rightarrow r_i(4) \]
The refinement of $\pi^u_\ast(MU_R^{(4)})$ (continued)

We will explain how $\pi^u_\ast(MU_R^{(4)})$ can be refined.

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$\pi_2(MU^{(4)}_R)$ has 4 generators $r_1(j)$ that are permuted up to sign by $G$. It is refined by an equivariant map

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Note that the slice cell $\hat{S}(\rho_2)$ is underlain by a wedge of 4 copies of $S^2$. 
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The refinement of $\pi_*^U(MU_R^{(4)})$ (continued)

In $\pi_*^U(MU_R^{(4)})$ there are 14 monomials that fall into 4 orbits (up to sign) under the action of $G$, each corresponding to a map from a $\hat{S}(m\rho_h)$.
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\begin{array}{c}
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$\hat{S}(2\rho_2) = C_{8+} \wedge C_2 \ S^{2\rho_2} \iff \{ r_1(1)^2, r_1(2)^2, r_1(3)^2, r_1(4)^2 \}$

Note that the slice cells $\hat{S}(2\rho_2)$ and $\hat{S}(\rho_4)$ are underlain by wedges of 4 and 2 copies of $S^4$ respectively.
The refinement of $\pi_{\ast}^{U}(MU_{R}^{(4)})$ (continued)

In $\pi_{4}^{U}(MU_{R}^{(4)})$ there are 14 monomials that fall into 4 orbits (up to sign) under the action of $G$, each corresponding to a map from a $\hat{S}(m\rho h)$.

\[\hat{S}(2\rho_{2}) = C_{8+} \wedge C_{2} \ S^{2\rho_{2}} \longleftrightarrow \{ r_{1}(1)^{2}, r_{1}(2)^{2}, r_{1}(3)^{2}, r_{1}(4)^{2} \} \]

\[\hat{S}(2\rho_{2}) \longleftrightarrow \{ r_{1}(1)r_{1}(2), r_{1}(2)r_{1}(3), r_{1}(3)r_{1}(4), r_{1}(4)r_{1}(1) \} \]
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Note that the slice cells $\hat{S}(2\rho_2)$ and $\hat{S}(\rho_4)$ are underlain by wedges of 4 and 2 copies of $S^4$ respectively.
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The refinement of $\pi_*^{MU}(MU_R^{(4)})$ (continued)

$$r_i(1) \rightarrow r_i(2) \rightarrow r_i(3) \rightarrow r_i(4)$$

(-1)^j

In $\pi_*^{MU}(MU_R^{(4)})$ there are 14 monomials that fall into 4 orbits (up to sign) under the action of $G$, each corresponding to a map from a $\hat{S}(m\rho_h)$.

$\hat{S}(2\rho_2) = C_{8+} \wedge C_2 S^{2\rho_2} \leftrightarrow \{r_1(1)^2, r_1(2)^2, r_1(3)^2, r_1(4)^2\}$

$\hat{S}(2\rho_2) \leftrightarrow \{r_1(1)r_1(2), r_1(2)r_1(3), r_1(3)r_1(4), r_1(4)r_1(1)\}$

$\hat{S}(2\rho_2) \leftrightarrow \{r_2(1), r_2(2), r_2(3), r_2(4)\}$

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Note that the slice cells $\hat{S}(2\rho_2)$ and $\hat{S}(\rho_4)$ are underlain by wedges of 4 and 2 copies of $S^4$ respectively.
The refinement of $\pi_*(MU_R^{(4)})$ (continued)

\[ (-1)^j \]

\[ r_i(1) \to r_i(2) \to r_i(3) \to r_i(4) \]
The refinement of $\pi_*^u(MU_R^{(4)})$ (continued)

It follows that $\pi_*^u(MU_R^{(4)})$ is refined by an equivariant map from

$$\hat{W}_2 = \hat{S}(2\rho_2) \lor \hat{S}(2\rho_2) \lor \hat{S}(2\rho_2) \lor \hat{S}(\rho_4).$$
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A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties.
The slice spectral sequence (continued)

Slice Theorem

In the slice tower for $\text{MU}_R^{(g/2)}$, every odd slice is contractible.
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In the slice tower for $MU^{(g/2)}_R$, every odd slice is contractible, and the $2m$th slice is $\hat{W}_m \wedge H\mathbb{Z}$, where $\hat{W}_m$ is the wedge of slice cells indicated above and $H\mathbb{Z}$ is the integer Eilenberg-Mac Lane spectrum.
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This result is the technical heart of our proof.

Thus we need to find the groups

$$\pi^G_\ast(\hat{S}(m\rho h) \wedge H\mathbb{Z}) = \pi^H_\ast(S^{m\rho h} \wedge H\mathbb{Z}) = \pi_\ast((S^{m\rho h} \wedge H\mathbb{Z})^H).$$
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In the slice tower for $\text{MU}_R^{(g/2)}$, every odd slice is contractible, and the $2m$th slice is $\hat{W}_m \wedge \mathbb{H}$, where $\hat{W}_m$ is the wedge of slice cells indicated above and $\mathbb{H}$ is the integer Eilenberg-Mac Lane spectrum. $\hat{W}_m$ never has any free summands.

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$$\pi_*^G(\hat{S}(m\rho_h) \wedge \mathbb{H}) = \pi_*^H(S^{m\rho_h} \wedge \mathbb{H}) = \pi_*((S^{m\rho_h} \wedge \mathbb{H})^H).$$

We need this for all nontrivial subgroups $H$ and all integers $m$ because we construct the spectrum $\hat{\Omega}$ by inverting a certain element in $\pi^G_{19\rho_8}(\text{MU}_R^{(4)})$. Here is what we will learn.
Computing $\pi^G_\ast(W(m\rho h) \wedge HZ)$

Vanishing Theorem

- For $m \geq 0$, $\pi^H_k(S^{m\rho h} \wedge HZ) = 0$ unless $m \leq k \leq hm$. 
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Gap Corollary

For $h > 1$ and all integers $m$, $\pi^H_k(S^{m\rho h} \wedge H\mathbb{Z}) = 0$ for $-4 < k < 0$. 

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Given the Slice Theorem, this means a similar statement must hold for $\pi^C\mathbb{Z}_*(\tilde{\Omega}) = \pi^\ast_\mathbb{Z}(\Omega)$, which gives the Gap Theorem.
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Computing $\pi_*^G(W(m\rho h) \wedge H\mathbb{Z})$ (continued)

Here again is a picture showing $\pi_*^{C_8}(S^{m\rho_8} \wedge H\mathbb{Z})$ for small $m$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{picture.png}
\end{figure}
Computing $\pi_*^G(W(m \rho h) \wedge HZ)$ (continued)

Here again is a picture showing $\pi_*^C(S^{m \rho 8} \wedge HZ)$ for small $m$. 
The proof of the Gap Theorem

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and so on.
The proof of the Gap Theorem (continued)

In other words,

\[
C(m \rho g)_k = \begin{cases} 
0 & \text{unless } m \leq k \leq gm \\
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where \( G' \) and \( G'' \) are the subgroups of indices 2 and 4. Each of these is a cyclic \( \mathbb{Z}[G] \)-module. The boundary operator is uniquely determined by the fact that \( H_*(C(m\rho g)) = H_*(S^{gm}) \).
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Then we have

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\pi_*^G(S^{m_\rho g} \wedge H\mathbb{Z}) = H_*(\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, C(m_\rho g))) = H_*(C(m_\rho g)^G).
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These groups are nontrivial only for \( m \leq k \leq gm \), which gives the Vanishing Theorem for \( m \geq 0 \).
The proof of the Gap Theorem (continued)

We will look at the bottom three groups in the complex \( \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, C(m\rho g)_*) \).
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For $m > 1$ our chain complex $C(m_{\rho g})$ has the form

$$
\begin{align*}
0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}[C_2] \leftarrow \mathbb{Z}[C_2] \leftarrow \cdots
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We will look at the bottom three groups in the complex \( \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, C(m\rho_g)_*) \). Since \( C(m\rho_g)_k \) is a cyclic \( \mathbb{Z}[G] \)-module, the Hom group is always \( \mathbb{Z} \).

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\[
\begin{array}{cccc}
C(m\rho_g)_m & C(m\rho_g)_{m+1} & C(m\rho_g)_{m+2} \\
\downarrow & \downarrow 1-\gamma & \downarrow 1+\gamma \\
0 & \mathbb{Z} & \mathbb{Z}[C_2]
\end{array}
\]

Applying \( \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot) \) (taking fixed points) to this gives (in dimensions \( \leq 2m \) for \( m > 4 \))

\[
\begin{array}{cccc}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
2 & 0 & 2 & 0 \\
m & m+1 & m+2 & m+3 & m+4
\end{array}
\]
Again, $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, C(m\rho g))$ in low dimensions is

\[
\begin{array}{cccccc}
\mathbb{Z} & \overset{2}{\leftarrow} & \mathbb{Z} & \overset{0}{\leftarrow} & \mathbb{Z} & \overset{2}{\leftarrow} & \mathbb{Z} & \overset{0}{\leftarrow} & \mathbb{Z} & \overset{2}{\leftarrow} & \mathbb{Z} & \leftarrow \cdots \\
m & m + 1 & m + 2 & m + 3 & m + 4
\end{array}
\]

It follows that for $m \leq k < 2m$,

$$
\pi_{G,k}(S_{m\rho g} \wedge H\mathbb{Z}) = \begin{cases} 
\mathbb{Z}/2 & \text{if } k \equiv m \mod 2 \\
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$$
The proof of the Gap Theorem (continued)

Again, $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, C(m\rho_g))$ in low dimensions is

$$\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow \ldots$$

$$m \quad m + 1 \quad m + 2 \quad m + 3 \quad m + 4$$

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We can study the groups $\pi_*^G(S^{mpg} \wedge HZ)$ for $m < 0$ in two different ways, topologically and algebraically.
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We can study the groups $\pi_*^G(S^{m\rho g} \wedge H\mathbb{Z})$ for $m < 0$ in two different ways, topologically and algebraically.

For the topological approach, it is the same as the graded group

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Since $G$ acts trivially on the target $HZ$, equivariant maps to it are the same as ordinary maps from the orbit space $S^{-m \rho g}/G$. 
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where $m < 0$.

Since $G$ acts trivially on the target $H\mathbb{Z}$, equivariant maps to it are the same as ordinary maps from the orbit space $S^{-m\rho g} / G$.

For simplicity, assume that $G = C_2$. 
The proof of the Gap Theorem (continued)

We can study the groups $\pi_*(S^{m \rho g} \wedge HZ)$ for $m < 0$ in two different ways, topologically and algebraically.

For the topological approach, it is the same as the graded group

$$[S^{-m \rho g}, HZ]^G_*$$

where $m < 0$.

Since $G$ acts trivially on the target $HZ$, equivariant maps to it are the same as ordinary maps from the orbit space $S^{-m \rho g}/G$.

For simplicity, assume that $G = C_2$. Then the orbit space is $\Sigma^{-m+1} \mathbb{RP}^{-m-1}$, and we are computing its ordinary reduced cohomology with integer coefficients.
The proof of the Gap Theorem (continued)

We can study the groups $\pi^G_\ast(S^{m\rho g} \wedge HZ)$ for $m < 0$ in two different ways, topologically and algebraically.

For the topological approach, it is the same as the graded group

$$[S^{-m\rho g}, HZ]^*_G$$

where $m < 0$.

Since $G$ acts trivially on the target $HZ$, equivariant maps to it are the same as ordinary maps from the orbit space $S^{-m\rho g}/G$.

For simplicity, assume that $G = C_2$. Then the orbit space is $\Sigma^{-m+1}RP^{-m-1}$, and we are computing its ordinary reduced cohomology with integer coefficients. We have

$$\pi^G_{-k}(S^{m\rho g} \wedge HZ)$$

$$= \overline{H}^k(\Sigma^{-m+1}RP^{-m-1}; Z)$$
The proof of the Gap Theorem (continued)

We can study the groups $\pi^G_\ast(S^{m\rho g} \wedge H\mathbb{Z})$ for $m < 0$ in two different ways, topologically and algebraically.

For the topological approach, it is the same as the graded group

$$[S^{-m\rho g}, H\mathbb{Z}]^G_*$$

where $m < 0$.

Since $G$ acts trivially on the target $H\mathbb{Z}$, equivariant maps to it are the same as ordinary maps from the orbit space $S^{-m\rho g} / G$.

For simplicity, assume that $G = C_2$. Then the orbit space is $\Sigma^{−m+1} \mathbb{R}P^{-m-1}$, and we are computing its ordinary reduced cohomology with integer coefficients. We have

$$\pi^G_{-k}(S^{m\rho g} \wedge H\mathbb{Z})$$

$$= \overline{H}^k(\Sigma^{-m+1} \mathbb{R}P^{-m-1} ; \mathbb{Z})$$

$$= 0 \left\{ \begin{array}{l} \text{unless } k = −m + 2 \text{ when } m = −2 \\ \text{unless } −m + 3 \leq k \leq −2m \text{ when } m \leq −3. \end{array} \right.$$
The proof of the Gap Theorem (continued)

We can study the groups $\pi_\ast^G(S^{m\rho g} \wedge H\mathbb{Z})$ for $m < 0$ in two different ways, topologically and algebraically.

For the topological approach, it is the same as the graded group

$$[S^{-m\rho g}, H\mathbb{Z}]_\ast^G$$

where $m < 0$.

Since $G$ acts trivially on the target $H\mathbb{Z}$, equivariant maps to it are the same as ordinary maps from the orbit space $S^{-m\rho g} / G$.

For simplicity, assume that $G = C_2$. Then the orbit space is $\Sigma^{-m+1}\mathbb{R}P^{-m-1}$, and we are computing its ordinary reduced cohomology with integer coefficients. We have

$$\pi_{-k}^G(S^{m\rho g} \wedge H\mathbb{Z})$$

$$= H^k(\Sigma^{-m+1}\mathbb{R}P^{-m-1}; \mathbb{Z})$$

$$= 0 \left\{ \begin{array}{ll}
\text{unless } k = -m + 2 \text{ when } m = -2 \\
\text{unless } -m + 3 \leq k \leq -2m \text{ when } m \leq -3.
\end{array} \right.$$ 

The increased lower bound is responsible for the gap.
The proof of the Gap Theorem (continued)

Alternatively, $S^{m \rho g}$ (with $m < 0$) is the equivariant Spanier-Whitehead dual of $S^{-m \rho g}$.
The proof of the Gap Theorem (continued)

Alternatively, \( S^{m\rho g} \) (with \( m < 0 \)) is the equivariant Spanier-Whitehead dual of \( S^{-m\rho g} \). This means that

\[
\pi_*^G(S^{m\rho g} \wedge H\mathbb{Z}) = H^*(\text{Hom}_{\mathbb{Z}[G]}(C(-m\rho g), \mathbb{Z})).
\]
The proof of the Gap Theorem (continued)

Alternatively, $S^{m \rho g}$ (with $m < 0$) is the equivariant Spanier-Whitehead dual of $S^{-m \rho g}$. This means that

$$\pi^G_\ast(S^{m \rho g} \wedge H\mathbb{Z}) = H^\ast(\text{Hom}_{\mathbb{Z}[G]}(C(-m \rho g), \mathbb{Z})).$$

Applying the functor $\text{Hom}_{\mathbb{Z}[G]}(\cdot, \mathbb{Z})$ to our chain complex $C(-m \rho g)$

$$\mathbb{Z} \xleftarrow{\epsilon} \mathbb{Z}[C_2] \xleftarrow{1-\gamma} \mathbb{Z}[C_2] \xleftarrow{1+\gamma} \mathbb{Z}[C_2 \text{ or } C_4] \xleftarrow{1-\gamma} \ldots$$

$-m \quad -m + 1 \quad -m + 2 \quad -m + 3$
The proof of the Gap Theorem (continued)

Alternatively, $S^{m \rho g}$ (with $m < 0$) is the equivariant Spanier-Whitehead dual of $S^{-m \rho g}$. This means that

$$
\pi_*^G(S^{m \rho g} \wedge H\mathbb{Z}) = H^*(\text{Hom}_{\mathbb{Z}[G]}(C(-m \rho g), \mathbb{Z})).
$$

Applying the functor $\text{Hom}_{\mathbb{Z}[G]}(\cdot, \mathbb{Z})$ to our chain complex $C(-m \rho g)$

\[
\begin{align*}
\mathbb{Z} & \xleftarrow{\epsilon} \mathbb{Z}[C_2] & \xleftarrow{1-\gamma} \mathbb{Z}[C_2] & \xleftarrow{1+\gamma} \mathbb{Z}[C_2 \text{ or } C_4] & \xleftarrow{1-\gamma} \cdots \\
-m & \quad -m+1 & \quad -m+2 & \quad -m+3
\end{align*}
\]

gives a negative dimensional chain complex beginning with

\[
\begin{align*}
\mathbb{Z} & \overset{1}{\rightarrow} \mathbb{Z} & \overset{0}{\rightarrow} \mathbb{Z} & \overset{2}{\rightarrow} \mathbb{Z} & \overset{0}{\rightarrow} \mathbb{Z} & \cdots \\
m & \quad m-1 & \quad m-2 & \quad m-3 & \quad m-4
\end{align*}
\]
The proof of the Gap Theorem (continued)

Here is a diagram showing both functors in the case $m \leq -4$. 

![Diagram showing functors in the case $m \leq -4$.]
The proof of the Gap Theorem (continued)

Here is a diagram showing both functors in the case $m \leq -4$.

$\begin{array}{cccccc}
-m & -m+1 & -m+2 & -m+3 & -m+4 \\
\mathbb{Z} & \mathbb{Z}[C_2] & \mathbb{Z}[C_2] & \mathbb{Z}[C_2] & \mathbb{Z}[C_2] & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots \\
\end{array}$

Note the difference in behavior of the map $\epsilon: \mathbb{Z}[C_2] \to \mathbb{Z}$ under the functors $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot)$ and $\text{Hom}_{\mathbb{Z}[G]}(\cdot, \mathbb{Z})$. They convert it to maps of degrees 2 and 1 respectively. This difference is responsible for the gap.
The proof of the Gap Theorem (continued)

Here is a diagram showing both functors in the case $m \leq -4$.

$$
\begin{array}{ccccccc}
-m & -m + 1 & -m + 2 & -m + 3 & -m + 4 \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \ldots \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \ldots \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \ldots \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \ldots \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \ldots \\
\end{array}
$$

Note the difference in behavior of the map $\epsilon : \mathbb{Z}[C_2] \to \mathbb{Z}$ under the functors $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot)$ and $\text{Hom}_{\mathbb{Z}[G]}(\cdot, \mathbb{Z})$. 
The proof of the Gap Theorem (continued)

Here is a diagram showing both functors in the case \( m \leq -4 \).

\[
\begin{array}{ccccccc}
-m & -m+1 & -m+2 & -m+3 & -m+4 \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots \\
\mathbb{Z} & \mathbb{Z}[C_2] & \mathbb{Z}[C_2] & \mathbb{Z}[C_2] & \mathbb{Z}[C_2] & \cdots \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots \\
\end{array}
\]

Note the difference in behavior of the map \( \epsilon : \mathbb{Z}[C_2] \to \mathbb{Z} \) under the functors \( \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot) \) and \( \text{Hom}_{\mathbb{Z}[G]}(\cdot, \mathbb{Z}) \). They convert it to maps of degrees 2 and 1 respectively.
The proof of the Gap Theorem (continued)

Here is a diagram showing both functors in the case $m \leq -4$.

\[
\begin{array}{cccccc}
-m & -m + 1 & -m + 2 & -m + 3 & -m + 4 \\
Z & Z & Z & Z & Z & \cdots \\
& 2 & 0 & 2 & 0 & \\
\end{array}
\]

\[
\begin{array}{cccccc}
n & n - 1 & n - 2 & n - 3 & n - 4 \\
Z & Z & Z & Z & Z & \cdots \\
& 1 & 0 & 2 & 0 & \\
\end{array}
\]

Note the difference in behavior of the map $\epsilon : \mathbb{Z}[C_2] \rightarrow \mathbb{Z}$ under the functors $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot)$ and $\text{Hom}_{\mathbb{Z}[G]}(\cdot, \mathbb{Z})$. They convert it to maps of degrees 2 and 1 respectively. This difference is responsible for the gap.
A homotopy fixed point spectral sequence

Our strategy
The main theorem
How we construct $\Omega$

$MU$
Basic properties
Refining homotopy

Proof of Gap Theorem
The corresponding slice spectral sequence