1 \( \theta_j \) in the Adams-Novikov spectral sequence

Browder’s theorem says that \( \theta_j \) is detected in the classical Adams spectral sequence by

\[
h_j^2 \in \text{Ext}^{2,2j+1}_A(\mathbb{Z}/2,\mathbb{Z}/2).
\]

This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

\[
\beta_i/j \in \text{Ext}^{2,6i-2j}_{MU_*(MU)}(MU_*,MU_*)
\]

for certain values of \( i \) and \( j \). When \( j = 1 \), it is customary to omit it from the notation.

Here are the first few of these in the relevant bidegrees.

- \( \theta_4 : \beta_{8/8} \) and \( \beta_{6/2} \)
- \( \theta_5 : \beta_{16/16}, \beta_{12/4} \) and \( \beta_{11} \)
- \( \theta_6 : \beta_{32/32}, \beta_{24/8} \) and \( \beta_{22/2} \)
- \( \theta_7 : \beta_{64/64}, \beta_{48/16}, \beta_{44/4} \) and \( \beta_{43} \)

and so on. In the bidegree of \( \theta_j \), only \( \beta_{2j-1/2j-1} \) has a nontrivial image (namely \( h_j^2 \)) in the Adams spectral sequence. There is an additional element in this bidegree, namely \( \alpha_1 \alpha_2 \).

We need to show that any element mapping to \( h_j^2 \) in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for \( \Omega \).

Detection Theorem. Let \( x \in \text{Ext}^{2,2j+1}_{MU_*(MU)}(MU_*,MU_*) \) be any element whose image in \( \text{Ext}^{2,2j+1}_A(\mathbb{Z}/2,\mathbb{Z}/2) \) is \( h_j^2 \) with \( j \geq 6 \). (Here \( A \) denotes the mod 2 Steenrod algebra.) Then the image of \( x \) in \( H^{2,2j+1}(C_8;\pi_*(\tilde{\Omega})) \) is nonzero.

We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, the theory of formal \( A \)-modules, where \( A \) is the ring of integers in a suitable field.
2 Formal $A$-modules

Formal $A$-modules

Recall the formal group law over a ring $R$ is a power series

$$F(x, y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x,y]]$$

with certain properties.

For positive integers $m$ one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m+n](x) = F([m](x), [n](x)) \quad \text{and} \quad [m][n](x) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers $m$, and we get a homomorphism $\tau$ from $\mathbb{Z}$ to $\text{End}(F)$, the endomorphism ring of $F$.

Formal $A$-modules (continued)

If the ground ring $R$ is an algebra over the $p$-local integers $\mathbb{Z}((p))$ or the $p$-adic integers $\mathbb{Z}_p$, then we can make sense of $[m](x)$ for $m$ in $\mathbb{Z}((p))$ or $\mathbb{Z}_p$.

Now suppose $R$ is an algebra over a larger ring $A$, such as the ring of integers in a number field or a finite extension of the $p$-adic numbers. We say that the formal group law $F$ is a formal $A$-module if the homomorphism $\tau$ extends to $A$ in such a way that

$$[a](x) \equiv ax \mod (x^2) \quad \text{for} \quad a \in A.$$

The theory of formal $A$-modules is well developed. Lubin-Tate used them to do local class field theory.

Formal $A$-modules (continued)

The example of interest to us is $A = \mathbb{Z}_2[x]/(x^4 + 1) = \mathbb{Z}_2[\zeta_8]$, where $\zeta_8$ is a primitive 8th root of unity. The maximal ideal of $A$ is generated by $\pi = \zeta_8 - 1$, and $\pi^4$ is a unit multiple of 2. There is a formal $A$-module $G$ over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) satisfying

$$\log_G(G(x,y)) = \log_G(x) + \log_G(y)$$

where

$$\log_G(x) = \sum_{n \geq 0} \frac{w^{2^n-1}x^{2^n}}{\pi^{2^n}}.$$

The classifying map $\lambda : MU_* \to R_*$ for $G$ factors through $BP_*$, where the logarithm is

$$\log_P(x) = \sum_{n \geq 0} \ell_n x^{2^n}.$$
Formal $\Lambda$-modules (continued)

Recall that $BP_* = \mathbb{Z}_2[v_1, v_2, \ldots]$ with $|v_n| = 2(2^n - 1)$. The $v_n$ and the $\ell_n$ are related by Hazewinkel’s formula,

\[
\begin{align*}
\ell_1 &= \frac{v_1}{2} \\
\ell_2 &= \frac{v_2}{2} + \frac{v_3}{4} \\
\ell_3 &= \frac{v_3}{2} + \frac{v_1v_2 + v_2v_1^2}{4} + \frac{v_2}{8} \\
\ell_4 &= \frac{v_4}{2} + \frac{v_1v_3 + v_2 + v_3v_1^2}{4} + \frac{v_3v_2 + v_2v_1^2 + v_2v_1^2 + v_2v_1^2}{8} + \frac{v_3}{16} \\
&\vdots
\end{align*}
\]

3 $\pi_\ast(MU^{(4)})$ and $R_*$

The relation between $MU^{(4)}$ and formal $\Lambda$-modules

What does all this have to do with our spectrum $\tilde{\Omega} = D^{-1}MU^{(4)}$? Recall that $D = \Delta_1^{(8)}\Delta_2^{(4)}\Delta_2^{(4)}\Delta_4^{(2)}$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\Delta$. They are the smallest ones that satisfy the second part of the following.

Lemma. The classifying homomorphism $\lambda : \pi_\ast(MU) \to R_*$ for $G$ factors through $\pi_\ast(MU^{(4)})$ in such a way that

- the homomorphism $\lambda^{(4)} : \pi_\ast(MU^{(4)}) \to R_*$ is equivariant, where $C_8$ acts on $\pi_\ast(MU^{(4)})$ as before, it acts trivially on $A$ and $\gamma w = \xi_8 w$ for a generator $\gamma$ of $C_8$.

- The element $D \in \pi_\ast(MU^{(4)})$ that we invert to get $\tilde{\Omega}$ goes to a unit in $R_*$.

We will prove this later.

4 The proof of the Detection Theorem

The proof of the Detection Theorem

It follows that we have a map

$$H^\ast(C_8; \pi_\ast(D^{-1}MU^{(4)})) = H^\ast(C_8; \pi_\ast(\tilde{\Omega})) \to H^\ast(C_8; R_*)$$

The source here is the $E_2$-term of the homotopy fixed point spectral sequence for $\pi_\ast(\Omega)$, and the target is easy to calculate. We will use it to prove the Detection Theorem, namely

Detection Theorem. Let $x \in \text{Ext}^{2,2j+1}_{MU^{(4)}(MU_\ast, MU_\ast)}$ be any element whose image in $\text{Ext}^{2,2j+1}_A(\mathbb{Z}/2, \mathbb{Z}/2)$ is $h_7^2$ with $j \geq 6$. (Here $A$ denotes the mod 2 Steenrod algebra.) Then the image of $x$ in $H^{2,2j+1}(C_8; \pi_\ast(\tilde{\Omega}))$ is nonzero.

We will prove this by showing that the image of $x$ in $H^{2,2j+1}(C_8; R_*)$ is nonzero.

The proof of the Detection Theorem (continued)

We will calculate with $BP$-theory. Recall that

$$BP_* (BP) = BP_* [t_1, t_2, \ldots] \text{ where } |t_n| = 2(2^n - 1).$$

We will abbreviate $\text{Ext}_{BP_* (BP)}^{d,j} (BP_*, BP_*)$ by $\text{Ext}^{d,j}$. For a $BP_* (BP)$-comodule $M$ (such as $BP_* (X)$), we will abbreviate $\text{Ext}_{BP_* (BP)} (BP_*, BP_*)$ by $\text{Ext}(M)$.
There is a map from this Hopf algebroid to one associated with $H^*(C_8; R_s)$ in which $t_s$ maps to an $R_s$-valued function on $C_8$ (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{n \geq 0} \langle t_n, \zeta \rangle x^n.$$  

An easy calculation shows that the function $t_1$ sends a primitive root in $C_8$ to a unit in $R_s$.

The proof of the Detection Theorem (continued)

Let

$$b_{1,j-1} = \frac{1}{2} \sum_{0<i<2j} \binom{2j}{i} [t_i^j t_i^{2j-i}] \in \text{Ext}^{2,2j-1}$$

It is is known to be cohomologous to $\beta_{2j-1/2j-1}$ and to have order 2. We will show that its image in $H^{2,2j-1}(C_8; R_s)$ is nontrivial for $j \geq 2$.

$H^*(C_8; R_s)$ is the cohomology of the cochain complex

$$R_s[C_8] \xrightarrow{\gamma^1} R_s[C_8] \xrightarrow{\text{Trace}} R_s[C_8] \xrightarrow{\gamma^1} \cdots$$

where Trace is multiplication by $1 + \gamma + \cdots + \gamma^7$.

The proof of the Detection Theorem (continued)

The cohomology groups $H^*(C_8; R_s)$ for $s > 0$ are periodic in $s$ with period 2. We have

$$H^1(C_8; R_{2m}) = \ker (1 + \sum \xi^m)/\ker (\xi^m - 1)$$

$$= \left\{ \begin{array}{ll}
    w^m A/(\pi) & \text{for } m \text{ odd} \\
    w^m A/(\pi^2) & \text{for } m \equiv 2 \text{ mod } 4 \\
    w^m A/(2) & \text{for } m \equiv 4 \text{ mod } 8 \\
    0 & \text{for } m \equiv 0 \text{ mod } 8
  \end{array} \right.$$  

$$H^2(C_8; R_{2m}) = \ker (\xi^m - 1)/\ker (1 + \sum \xi^m)$$

$$= \left\{ \begin{array}{ll}
    w^m A/(8) & \text{for } m \equiv 0 \text{ mod } 8 \\
    0 & \text{otherwise}
  \end{array} \right.$$  

An easy calculation shows that $b_{1,j-1}$ maps to $4w^{2j}$, which is the element of order 2 in $H^2(C_8; R_{2j+1})$.

Sidebar on chromatic fractions

It is common to write $\beta_{i/j}$ as a chromatic fraction $\frac{\gamma^i}{2v^j}$. What does this mean? For suitable $i$ and $j$, $v^j$ is an element of $\text{Ext}^{0,6i}(BP_s/(2, v^j))$ and there are short exact sequences

and

$$0 \longrightarrow BP_s/(2) \xrightarrow{v^j/2} BP_s/(2, v^j) \longrightarrow 0$$

leading to connecting homomorphisms

$$\text{Ext}^{0,6i}(BP_s/(2, v^j)) \xrightarrow{v^j/2v^j} \text{Ext}^{1,6i-2j}(BP_s/(2)) \xrightarrow{v^j/2v^j} \text{Ext}^{2,6i-2j}(BP_s).$$
The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree as $\beta_{(j,0)/2j-1} = \beta_{c(j,0)/2j-1}$ map to zero. We will do this for $j \geq 6$. The set of these is

$$\left\{ \beta_{c(j,k)/2j-1-2k} : 0 < k < j/2 \right\}$$

where $c(j,k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$.

We will see in the proof of the Lemma below that $v_1$ and $v_2$ map to unit multiples of $\pi^3 w$ and $\pi^2 w^3$ respectively. This means we can define a valuation on chromatic fractions compatible with the one on $A$ in which $\|2\| = 1$, $\|\pi\| = 1/4$, $\|v_1\| = 3/4$ and $\|v_2\| = 1/2$. We extend the valuation on $A$ to $R_*$ by setting $\|w\| = 0$.

The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$\|\beta_{c(j,k)/2j-1-2k}\| = \left\| \frac{v_2^{c(j,k)}}{2v_2^{2j-1-2k}} \right\| = \frac{c(j,k)}{2} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 = \frac{2^j + 2^{j-1-2k}}{6} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 = \frac{(2^{j-1} - 7 \cdot 2^{j-3-2k})}{3} - 1 \geq 5.$$ 

This means $\beta_{c(j,k)/2j-1-2k}$ maps to an element that is divisible by 8 and therefore zero, since the homomorphism cannot lower this valuation.

The proof of the Detection Theorem (continued)

We have to make a similar computation with the element $\alpha_1 \alpha_{2j-1}$. We have

$$\|\alpha_{2j-1}\| = \left\| \frac{v_2^{2j-1}}{2} \right\| = \frac{3(2^j - 1)}{4} - 1 \geq \frac{21}{4} - 1 \geq 4 \quad \text{for} \quad j \geq 3.$$ 

This completes the proof of the Detection Theorem modulo the Lemma.

5 The proof of the Lemma

The proof of the Lemma

Here it is again.

Lemma. The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for $G$ factors through $\pi_*(MU^{(4)})$ in such a way that

- the homomorphism $\lambda^{(4)} : \pi_*(MU^{(4)}) \to R_*$ is equivariant, where $C_8$ acts on $\pi_*(MU^{(4)})$ as before, it acts trivially on $A$ and $\gamma w = \zeta_8 w$ for a generator $\gamma$ of $C_8$.
- The element $D \in \pi_*(MU^{(4)})$ that we invert to get $\tilde{\Omega}$ goes to a unit in $R_*$. 

5
The proof of the Lemma (continued)
To prove the first part, consider the following diagram for an arbitrary ring $K$.

\[
\begin{array}{ccc}
\pi_*(MU) & \xrightarrow{\lambda_1} & K \\
\pi_*(MU) & \xrightarrow{\lambda_2} & K \\
\pi_*(MU) & \xrightarrow{\lambda_3} & K \\
\pi_*(MU) & \xrightarrow{\lambda_4} & K \\
\end{array}
\]

The maps $\lambda_1$ and $\lambda_2$ classify two formal group laws $F_1$ and $F_2$ over $K$. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a compatible strict isomorphism between $F_1$ and $F_2$.

The proof of the Lemma (continued)
Similarly consider the diagram

\[
\begin{array}{ccc}
\pi_*(MU) & \xrightarrow{\lambda_1} & K \\
\pi_*(MU) & \xrightarrow{\lambda_2} & K \\
\pi_*(MU) & \xrightarrow{\lambda_3} & K \\
\pi_*(MU) & \xrightarrow{\lambda_4} & K \\
\end{array}
\]

The existence of $\lambda^{(4)}$ is equivalent to that of compatible strict isomorphisms between the four formal group laws $F_j$ classified by the $\lambda_j$.

The proof of the Lemma (continued)
Now suppose further that $K$ has a $C_8$-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined $C_8$-action on $MU^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending $x$ to its formal inverse on each of the $F_j$.

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbb{Z}[\zeta_8]$-module structure on each of the $F_j$, which are all isomorphic. Setting $K = R_*$ proves the first part of the Lemma.

The proof of the Lemma (continued)
Here is the Lemma again.

**Lemma.** The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for $G$ factors through $\pi_*(MU^{(4)})$ in such a way that

- the homomorphism $\lambda^{(4)} : \pi_*(MU^{(4)}) \to R_*$ is equivariant, where $C_8$ acts on $\pi_*(MU^{(4)})$ as before, it acts trivially on $A$ and $\gamma w = \zeta_8 w$ for a generator $\gamma$ of $C_8$.
- The element $D \in \pi_*(MU^{(4)})$ that we invert to get $\tilde{\Omega}$ goes to a unit in $R_*$. 

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The proof of the Lemma (continued)

For the second part, recall that \( D = \overline{A}_1^{(8)} N_2^{(4)}(\overline{A}_2^{(4)}) N_2^{(2)}(\overline{A}_4^{(2)}) \), where
\[
\overline{A}_k^{(g)} = \begin{cases} 
  x_{2^{k-1}} - 1 & \text{for } g = 2 \\
  N_2^{(g)}(r_{2^{k-1}}) & \text{otherwise.}
\end{cases}
\]

Since our formal \( A \)-module is 2-typical we can do the calculations using \( BP \) in place of \( MU \). Hence we can replace \( x_{2^{k-1}} \in \pi_* MU \) by \( v_{k} \in \pi_* BP \) and \( r_{2^{k-1}} \in \pi_* MU \wedge MU \) by \( t_{k} \in \pi_* BP \wedge BP \). We have \( \overline{A}_k^{(2)} = v_{k} \). Using Hazewinkel’s formula we find that
\[
\begin{align*}
v_1 & \mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w = \text{unit} \cdot \pi^3 w \\
v_2 & \mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 = \text{unit} \cdot \pi^2 w^3 \\
v_3 & \mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 = \text{unit} \cdot \pi w^7 \\
v_4 & \mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15}.
\end{align*}
\]
(where each unit is in \( A \)) so \( v_4 \) (but not \( v_n \) for \( n < 4 \)) and therefore \( N_2^{(8)}(\overline{A}_4^{(2)}) \) maps to a unit in \( R_* \).

The proof of the Lemma (continued)

We have \( \overline{A}_k^{(2)} = t_k \). We consider the equivariant composite
\[
BP_{n}^{(2)} \to BP_{n}^{(4)} \to R_*
\]
under which
\[
\eta_{R}(\ell_{n}) \mapsto \frac{\zeta_{8}w^{2^{n-1}}}{\pi^{n}}.
\]
Using the right unit formula we find that
\[
\begin{align*}
t_1 & \mapsto (\pi + 2)w = \text{unit} \cdot \pi w \\
t_2 & \mapsto (\pi^2 + 5\pi^2 + 9\pi + 5)w^3.
\end{align*}
\]
This means \( t_2 \) (but not \( t_1 \)) and therefore \( N_4^{(8)}(\overline{A}_2^{(4)}) \) maps to a unit in \( R_* \).

The proof of the Lemma (continued)

Finally, we have \( \overline{A}_n^{(8)} = \ell_{n}(1) \in BP_{n}^{(4)} \), where \( \ell_{n}(1) \) is the analog of \( r_{2^{n-1}}(1) \). Then we find
\[
\begin{align*}
\ell_{n}(1) & \mapsto \frac{w^{2^{n-1}}}{\pi^{n}} \\
\ell_{n}(2) & \mapsto \frac{(\zeta_{8}w)^{2^{n-1}}}{\pi^{n}}.
\end{align*}
\]
This implies
\[
\overline{A}_1^{(8)} = \ell_{1}(2) - \ell_{1}(1) \mapsto \frac{\zeta_{8}w - w}{\pi} = w.
\]
Thus we have shown that each factor of
\[
D = \overline{A}_1^{(8)} N_4^{(8)}(\overline{A}_2^{(4)}) N_2^{(2)}(\overline{A}_4^{(2)})
\]
and hence \( D \) itself maps to a unit in \( R_* \), thus proving the lemma.