Toward higher chromatic analogs of elliptic cohomology

Douglas C. Ravenel
University of Rochester
January 17, 2003

A 1-dimensional formal group law over and ring R leads to a homomorphism (called a genus)

$$\varphi : \pi_*(MU) \to R$$

by Quillen's theorem. The functor

$$X \mapsto MU_*(X) \otimes_\varphi R$$

is a homology theory if certain algebraic conditions on φ are satisfied; this is the Landweber Exact Functor Theorem.

Suppose E is an elliptic curve defined over R. It is a 1-dimensional algebraic group, and choosing a local paramater at the identity leads to a formal group law $\widehat{E}$, the formal completion of E. Thus we can apply the machinery above and get an R-valued genus.

For example the *Jacobi quartic*, defined by the equation

$$y^2 = 1 - 2\delta x^2 + \epsilon x^4,$$

is an elliptic curve over the ring

$$R = \mathbf{Z}[1/2, \delta, \epsilon].$$

The resulting formal group law is

$$F(x, y) = \frac{x\sqrt{1 - 2\delta y^2 + \epsilon y^4} + y\sqrt{1 - 2\delta x^2 + \epsilon x^4}}{1 - \epsilon x^2 y^2};$$

this calculation is originally due to Euler. The resulting genus is known to satisfy Landweber's conditions (Landweber-Ravenel-Stong), and this leads to one definition of elliptic cohomology theory.

The rich structure of elliptic curves leads to interesting calcuations with the cohomology theory and the theory of topological modular forms due to Hopkins et al.

It is known that the formal group law associated with an elliptic curve over a finite field can have height at most 2. Hence elliptic cohomology cannot give us any information about v_n-periodic phenomena for $n > 2$.

QUESTION: How can we attach formal group laws of height > 2 to geometric objects (such as algebraic curves) and use them get insight into cohomology theories that go deeper into the chromatic tower?

PROGRAM:

- Let C be a curve of genus g over some ring R.
- Its Jacobian $J(C)$ is an abelian variety of dimension g.
- $J(C)$ has a formal completion $\widehat{J}(C)$ which is a g-dimensional formal group law.
- If $\widehat{J}(C)$ has a 1-dimensional summand, then Quillen's theorem gives us a genus associated with the curve C.

Theorem 1. *Let $C(p, f)$ be the Artin-Schreier curve over $\mathbf{F}_p$ defined by the affine equation*

$$y^d = x^p - x \qquad \textit{where } d = p^f - 1.$$

(Assume that $(p, f) \neq (2, 1)$.) Then its Jacobian $J(C(p, f))$ has a 1-dimensional formal summand of height $(p-1)f$.

Properties of $C(p, f)$:

- $C(p, f)$ is a d-fold branched covering of the projective line with $p + 1$ branch points.
- Its genus is $(p-1)(d-1)/2$, eg it is 0 in the excluded case, and 1 in the cases $(p, f) = (2, 2)$ and $(3, 1)$. In these cases C is an elliptic curve whose formal group law has height 2.
- Over $\mathbf{F}_{p^{(p-1)f}}$ it has an action by the group

$$G = \mathbf{F}_p \rtimes \mu_{(p-1)d}$$

given by

$$(x, y) \mapsto (\zeta^d x + a, \zeta y)$$

for $a \in \mathbf{F}_p$ and $\zeta \in \mu_{(p-1)d}$.

Remarks

- This result was known to and cited by Manin in 1963. Most of what is needed for the proof can be found in Katz's 1979 Bombay Colloquium paper and in Koblitz' Hanoi notes.
- The proof rests on the determination of the zeta function of the curve by Davenport-Hasse in 1934, and on some properties of Gauss sums proved by Stickelberger in 1890. The method leads to complete determination of $\widehat{J}(C(p,f))$.
- Let $\mathbf{G}_n$ denote the extension of the Morava stabilizer group S_n by the Galois group C_n. Given a finite subgroup $G \subset \mathbf{G}_n$, Hopkins-Miller can construct a "homotopy fixed point spectrum" E_n^{hG}. The group G from the previous page was shown by Hewett to be a maximal finite subgroup of $\mathbf{G}_n$ for $n = (p-1)f$. It acts on the 1-dimensional summand of $\widehat{J}(C(p,f))$ in the appropriate way.

More remarks

- Gorbunov-Mahowald studied this curve for $f = 1$. They found a lifting of the curve to characteristic zero associated with the Lubin-Tate lift of the formal group law of height $p - 1$.
- We are close to a new proof of the theorem based on Honda's theory of commutative formal group laws developed in the early '70s. It does not rely on knowledge of the zeta function and is thus a more promising approach to the lifting problem.

University of Rochester, Rochester, NY 14627