Theorem 1. Let \( C(p, f) \) be the Artin-Schreier curve over \( \mathbf{F}_p \) defined by the affine equation
\[
y^d = x^p - x \quad \text{where} \quad d = p^f - 1.
\]
(Assume that \( f > 1 \) when \( p = 2 \).) Then its Jacobian has a 1-dimensional formal summand of height \( (p - 1)f \).

Properties of \( C(p, f) \):
- Its genus is \( (p - 1)(d - 1)/2 \).
- It has an action by the group
  \[
  \tilde{G} = \mathbf{F}_p \rtimes \mu_{(p-1)d}
  \]
given by
  \[
  (x, y) \mapsto (\zeta^d x + a, \zeta y)
  \]
  for \( a \in \mathbf{F}_p \) and \( \zeta \in \mu_{(p-1)d} \).
- Its de Rham \( H^1 \) has basis
  \[
  \left\{ \omega_{i,j} = \frac{x^i y^j dx}{y^{d-1}} : 0 \leq i \leq p - 2, 0 \leq j \leq d - 2 \right\}.
  \]
- If we restrict the action to the abelian subgroup \( G = \mathbf{F}_p \times \mu_d \), \( H^1 \) decomposes into 1-dimensional eigenspaces for each character that is nontrivial on both \( \mathbf{F}_p \) and \( \mu_d \).
The Hopkins-Mahowald affine group action. The Weierstrass equation for a general elliptic curve is

\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]

Under the affine coordinate change

\[ x \mapsto x + r \quad \text{and} \quad y \mapsto y + sx + t \]

we get

\[ a_6 \mapsto a_6 + a_4 r + a_3 t + a_2 r^2 + a_1 r t + t^2 - r^3 \]

\[ a_4 \mapsto a_4 + a_3 s + 2a_2 r + a_1 (r s + t) + 2s t - 3r^2 \]

\[ a_3 \mapsto a_3 + a_1 r + 2t \]

\[ a_2 \mapsto a_2 + a_1 s - 3r + s^2 \]

\[ a_1 \mapsto a_1 + 2s. \]

This can be used to define an action of the affine group on the ring

\[ A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]. \]

Its cohomology is the \( E_2 \)-term of a spectral sequence converging to \( \pi_* \text{tmf} \).
Theorem 2. [Dieudonné] The category of formal groups over a finite field $k$ is equivalent to the category of modules over the ring

$$D(k) = W(k)(F, V)/(FV = VF = p)$$

where $Fw = w^\sigma F$ and $Vw^\sigma = wV$ for $w \in W(k)$. $F$ is the Frobenius or $p$th power map, and $V$ is the Verschiebung, the dual of $F$.

Examples:

- The Dieudonné module for the formal group law associated with the $n$th Morava K-theory is

  $$D(F_p)/(V - F^{n-1}),$$

  so in it we have $F^n = p$.

- More generally, for $m$ and $n$ relatively prime, let

  $$G_{m,n} = D(k)/(V^m - F^n).$$

  It corresponds to an $m$-dimensional formal group of height $m + n$. 
Theorem 3. [Manin]

(i) **Structure Theorem.** Any simple Dieudonné module $M$ is isogenous over $W(\overline{F}_p)$ to some $G_{m,n}$.

(ii) Let the characteristic polynomial for $F$ in $M$ be

$$Q(T) = T^m + \sum_{i>0} c_i T^{m-i}$$

for $c_i \in W(k)$. If its Newton polygon has a line segment of horizontal length $n$ and slope $j/n$, then up to isogeny over $W(\overline{k})$, $M$ has a summand of the form $G_{j,n-j}$.

The Newton polygon is the convex hull of the set of points

$$\{(i, \text{ord}_p(c_i)) : 0 \leq i \leq m\},$$

where $c_0 = 1$. The condition on $Q(T)$ above is equivalent to the existence of $n$ roots having $p$-adic valuation $j/n$. 
Theorem 4. [Manin, Tate, Honda]

(i) Riemann Symmetry Condition. If $A$ is an abelian variety with formal completion $\widehat{A}$, and its Dieudonné module $D(\widehat{A})$ has a summand $G_{m,n}$ up to isogeny over $W(F_p)$, then it also has a summand $G_{n,m}$.

(ii) More precisely, if $A$ has dimension $g$ and is defined over $F_q$ with $q = p^a$, then the characteristic polynomial for $F^a$ has the form

$$Q_a(T) = T^{2g} + \sum_{0 < i < 2g} c_i T^{2g-i} + q^g$$

with $c_i \in \mathbb{Z}$, and

$$Q_a \left( \frac{q}{T} \right) = \frac{q^g Q_a(T)}{T^{2g}}$$

so $c_{g+i} = q^i c_{g-i}$ for $0 < i < g$. (The Newton polygon for $Q(T)$ is determined by that of $Q_a(T)$.)

(iii) Classification of Abelian Varieties up to Isogeny over $F_q$. There is a one-to-one correspondence between isogeny classes of abelian varieties over $F_q$ and polynomials of the above form, all of whose roots have absolute value $\sqrt{q}$. 
Corollary 5.
(i) For an elliptic curve $C$, either
\[ D(\hat{C}) \cong G_{0,1} \oplus G_{1,0}, \]
(the ordinary height 1 case) or
\[ D(\hat{C}) \cong G_{1,1}, \]
(the supersingular height 2 case), up to isogeny over $W(F_p)$.
(ii) If an abelian variety $A$ has a 1-dimensional formal summand of height $n$ for $n > 2$, then the dimension of $A$ is at least $n$.

Theorem 6. [Grothendieck, Berthelot]
Let $C$ be a smooth curve of genus $g$ over $F_q$. Then its crystalline (or de Rham) $H^1$ is a free $W(F_q)$-module of rank $2g$ isomorphic to the Dieudonné module of its Jacobian $D(\hat{J}(C))$, with the induced action of the Frobenius $F$ relative to $F_q$ coinciding with the action of $F^a$. 

Given a smooth $d$-dimensional variety $X$ over $\mathbf{F}_q$, its Zeta function is defined by

$$Z(X, T) = \exp \left( \sum_{n>0} |X(\mathbf{F}_{q^n})| \frac{T^n}{n} \right).$$

Then

(i) $Z(X, T)$ is a rational function of $T$. (Proved by Dwork in 1960.)

(ii) More precisely,

$$Z(X, T) = \frac{P_1(T)P_3(T) \cdots P_{2d-1}(T)}{P_0(T)P_2(T) \cdots P_{2d}(T)}$$

where $P_i(T)$ is a polynomial whose degree is the rank of $H^i(X)$ suitably defined.

(iii) Riemann hypothesis in characteristic $p$. Each reciprocal root of $P_i(T)$ has absolute value $q^{i/2}$.

(iv)

$$P_i(T) = \det(1 - T\tilde{F}|H^i(X))$$

where $\tilde{F}$ is the Frobenius relative to $\mathbf{F}_q$. Hence (ii) follows from an analog of the Lefschetz fixed point formula.
Weil proved these statements for curves. If $X$ is a smooth curve of genus $g$, then
\[ Z(X, T) = \frac{P_1(T)}{(1 - T)(1 - qT)}, \]
where the factors $(1 - T)^{-1}$ and $(1 - qT)^{-1}$ correspond to $H^0$ and $H^2$. $P_1(T)$, which corresponds to $H^1$, has degree $2g$ with
\[ P_1(T) = 1 + \sum_{0 < i < 2g} c_i T^i + q^g T^{2g}, \]
and $Q_a(T) = T^{2g} P_1(1/T)$ is the characteristic polynomial of $\hat{F} = F^a$ in $D(\hat{J}(X))$. The coefficients $c_i$ are the same as those in Theorem 4.

In other words, the zeta function of a curve determines the formal structure of its Jacobian in an explicit way.
Suppose $X$ is acted on by a finite group $G$ and let $\rho$ be a representation of $G$ over a suitable number field $K$. Define

$L(X, \rho, T) = \exp \left( \frac{1}{|G|} \sum_{g \in G} \text{Trace}(\rho(g)) \sum_{n>0} C_n^g T^n \right)$,

where $C_n^g$ is the number of points in $x$ in $X(\overline{F}_p)$ satisfying $g(x) = \tilde{F}^n(x)$.

When $\rho$ is the regular representation, $L(X, \rho, T)$ is the zeta function. If the action of $G$ is trivial and $\rho$ is irreducible and nontrivial, then $L(X, \rho, T) = 1$.

We have

$L(X, \rho_1 \oplus \rho_2, T) = L(X, \rho_1, T)L(X, \rho_2, T)$

so

$Z(X, T) = \prod_{\rho \text{ irreducible}} L(X, \rho, T)^{\text{degree}(\rho)}$.

Deligne proved an alternating product formula for $L(X, \rho, T)$ similar to Weil’s for $Z(X, T)$, in which $P_i^\rho(T)$ is the characteristic polynomial of $\tilde{F}$ restricted to

$\text{Hom}_G(\rho, H^i(X) \otimes_{\text{W}(F_q)} K)$. 
Recall that our curve $C(p, f)$ admits an action of the abelian group
\[ G = \mathbb{F}_p \times \mu_d \]
where $d = p^f - 1$ that decomposes $H^1$ into 1-dimensional eigenspaces. It follows that
\[ P_1(T) = \prod_{\chi} P_1^\chi(T), \]
where the product is over all characters $\chi$ that are nontrivial on both factors of $G$. Each of these factors of $P_1(T)$ is linear. They were computed in 1935 by Davenport and Hasse, who showed that the reciprocal roots of $P_1(T)$ (which are the roots of $Q_f(T)$) are certain Gauss sums, i.e., sums of $pd$th roots of unity. They can be computed explicitly for small values of $p$ and $f$. The ideals that they generate, and hence their valuations with respect to a $p$-adic place in $K$, were determined by Stickelberger in 1890.
Theorem 7. The characteristic polynomial $Q(T)$ for the Frobenius in the Dieudonné module $D(\hat{J}(C'(p, f)))$ has $(p - 1)b_i$ roots with $p$-ordinal $i/(p - 1)$, where

$$
\sum_i b_i t^i = \left(\frac{1 - t^p}{1 - t}\right)^f - 1 - t^{(p-1)f}
$$

so for $0 < i < (p - 1)f$,

$$
b_i = \sum_{0 \leq j \leq i/p} (-1)^j \binom{f}{j} \binom{f + i - pj - 1}{f - 1},
$$
e.g., $b_1 = f$.

Theorem 1 and more is a corollary of this.

Corollary 8. In terms of Manin’s structure theorem,

$$
D(\hat{J}(C'(p, 1))) \cong \bigoplus_{0 < i < p - 1} G_{i, p-1-i}
$$

$$
D(\hat{J}(C'(p, 2))) \cong \binom{p}{2} G_{1, 1} \bigoplus \bigoplus_{0 < i < p - 1} \frac{i + 1}{2} (G_{i, 2p-2-i} \oplus G_{2p-2-i, i})
$$

$$
D(\hat{J}(C(2, f))) \cong \bigoplus_{0 < i < f} \binom{f}{i} \frac{1}{f} G_{i, f-i}
$$

up to isogeny, where it is understood that $G_{km, kn} = k G_{m, n}$. 
Here are some explicit values of the characteristic polynomial $Q(T)$ of the Frobenius (relative to $F_p$) for the curve $C(p, f)$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$f$</th>
<th>$Q(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>$T^2 + 2$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>$T^6 - 2T^3 + 2^3$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$(T^8 + 2T^4 + 2^4)(T^2 + 2T + 2)(T^2 - 2T + 2)(T^2 \pm 2)$</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>$T^{30} - 6T^{25} - 16T^{20} + 352T^{15} - 512T^{10} - 6144T^5 + 32768$</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>$(T^{36} + 6T^{30} + 120T^{24} + 384T^{18} + 7680T^{12} + 24576T^8 + 262144)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T^{12} - 12T^6 + 64)(T^{12} + 12T^6 + 64)(T^2 + 2)^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T^2 + 2)(T^4 - 2T^2 + 4)(T^2 + 8)$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$T^2 + 3$</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>$(T^8 - 6T^4 + 81)(T^2 - 3)^3(T^2 + 3)$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$(T^{24} - 87T^{18} + 3321T^{12} - 63423T^6 + 531441)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T^{12} + 9T^6 + 45T^4 + 243T^3 + 729)^2(T^2 + 3)$</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>$(T^8 + 30T^4 + 625)(T^2 - 5)^2$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>$(T^2 - 5)^8(T^2 + 5)^8(T^8 - 30T^4 + 625)^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T^8 + 30T^4 + 625)(T^{10} + 750T^4 + 390625)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T^{32} + 1380T^{24} + 1103750T^{16} + 539062500T^8 + 152587890625)$</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>$(T^{12} + 4977T^6 + 117649)(T^6 + 7T^3 + 343)^2(T^2 + 7)^3$</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>$(672749994932560009201 - 14568299213068271T^{10} + 129620301481T^{20} - 561671T^{30} + T^{40})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(25937424601 - 157668929T^5 + 467181T^{10} - 979T^{15} + T^{20})^2(T^2 + 11)^5$</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>$(542800770374370512771595361 - 415420467450868292270T^{12} + 120001160412387T^{24} - 17830670T^{36} + T^{48})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(4826809 + 4381T^6 + T^{12})^2(28561 - 130T^4 + T^8)^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2197 - 65T^9 + T^6)^4(-13 + T^2)^6$</td>
</tr>
</tbody>
</table>

Isaac Newton Institute, Cambridge, UK