A note on the thick subcategory theorem

Alain Jeanneret
University of Bern
CH-3012 Bern
jeanner@math-stat.unibe.ch

Peter S. Landweber *
Rutgers University
New Brunswick, New Jersey 08903 USA
landwebe@lagrange.rutgers.edu

Douglas C. Ravenel †
University of Rochester
Rochester, New York 14627 USA
drav@troi.cc.rochester.edu

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1 Introduction

In this paper we will discuss an algebraic version (Theorem 1.6) of the thick subcategory theorem of Hopkins-Smith [HS] (Theorem 1.4). The former is stated as Theorem 3.4.3 in [Rav92], but the proof given there is incorrect. (A list of errata for [Rav92] can be obtained by email from the third author.)

First we recall the nilpotence theorem in its p-local version. Let BP be the Brown-Peterson spectrum at the prime p, which satisfies:

$$\pi_*(BP) \cong BP_* \cong \mathbf{Z}_{(p)}[v_1, v_2, \cdots], \ \ |v_i| = 2(p^i - 1).$$

Theorem 1.1 (Nilpotence theorem) *[DHS88]*

(i) Let R be a p-local ring spectrum. The kernel of the BP Hurewicz homomorphism $BP_* : \pi_*(R) \longrightarrow BP_*(R)$ consists of nilpotent elements.

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(ii) *Let $f : F \longrightarrow X$ be a map from a p-local finite spectrum to an arbitrary spectrum. If $BP \wedge f$ is null homotopic, then f is smash nilpotent; i.e. the i-fold smash product $f^{(i)} = f \wedge \cdots \wedge f$ is null for i sufficiently large.*

(iii) *Let $\cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \longrightarrow \cdots$ be a sequence of p-local spectra with X_n c_n-connected. Suppose that $c_n \geq mn + b$ for some m and b. If $BP_* f_n = 0$ for all n then $\mathrm{hocolim} X_n$ is contractible.*

The Baas-Sullivan theory of bordism with singularities allows one to define ring spectra $K(n)$ and $P(n)$ for $0 < n < \infty$ satisfying [Rav86]:

$$\pi_*(K(n)) \cong K(n)_* \cong \mathbf{F}_p[v_n, v_n^{-1}]$$

$$\pi_*(P(n)) \cong P(n)_* \cong \mathbf{F}_p[v_n, v_{n+1}, \cdots]$$

as BP_*-algebras. We also set $P(0) = BP$ and $K(0) = H\mathbf{Q}$, the rational Eilenberg-Mac Lane spectrum. $K(n)$ is known as the n^{th} Morava K-theory at the prime p. The following corollary of the nilpotence theorem will be proved in §2. This is stated in [Rav92] as Corollary 5.1.5, but again the proof given there is incorrect.

Corollary 1.2 *Let W, X and Y be p-local finite spectra and $f : X \longrightarrow Y$. Then $W \wedge f^{(k)}$ is null homotopic for $k \gg 0$ if $K(n)_*(W \wedge f) = 0$ for all $n \geq 0$.*

Now let $\mathcal{CP}_0$ be the homotopy category of finite p-local spectra and let $\mathcal{CP}_n \subset \mathcal{CP}_0$ be the full subcategory of $K(n-1)_*$-acyclics. In [Rav84] it was shown that the $\mathcal{CP}_n$ fit into a sequence:

$$\cdots \subset \mathcal{CP}_{n+1} \subset \mathcal{CP}_n \subset \cdots \subset \mathcal{CP}_0.$$

Moreover all the inclusions are strict [Mit85].

Definition 1.3 *A full subcategory $\mathcal{C}$ of $\mathcal{CP}_0$ is* **thick** *if:*

(i) *An object weakly equivalent to an object in $\mathcal{C}$ is in $\mathcal{C}$.*

(ii) *If $X \longrightarrow Y \longrightarrow Z$ is a cofibration in $\mathcal{CP}_0$ and two of $\{X, Y, Z\}$ are in $\mathcal{C}$ then so is the third.*

(iii) *A retract of an object in $\mathcal{C}$ is in $\mathcal{C}$.*

Corollary 1.2 is the form of the nilpotence theorem needed to prove the thick subcategory theorem (see §5.3 of [Rav92]):

Theorem 1.4 (Thick subcategory theorem) *If $\mathcal{C}$ is a thick subcategory of $\mathcal{CP}_0$, then there exists an integer k such that $\mathcal{C} = \mathcal{CP}_k$.*

Before we state an algebraic version of Theorem 1.4 let us fix some notation. Let $\mathcal{BP}_0$ be the abelian category of $BP_*(BP)$-comodules finitely presented as BP_*-module [Lan76]. A typical object in $\mathcal{BP}_0$ is $BP_*(X)$ for X in $\mathcal{CP}_0$. We denote by $\mathcal{BP}_k$ the full subcategory of $\mathcal{BP}_0$ whose objects M satisfy $v_{k-1}^{-1}M = 0$ (we set $v_0 = p$). Results of Johnson-Yosimura [JY80] (see also [Lan79] for a more algebraic proof) show that:

$$\cdots \subset \mathcal{BP}_{k+1} \subset \mathcal{BP}_k \subset \cdots \subset \mathcal{BP}_0.$$

Definition 1.5 *Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is* **thick** *if it satisfies the following condition:*
If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$, M belongs to $\mathcal{C}$ if and only if M' and M'' belong to $\mathcal{C}$. (It means that $\mathcal{C}$ is stable under subobjects, quotient objects and extensions.)

The classification of the thick subcategories of $\mathcal{BP}_0$ is now the following; see §3 for the proof.

Theorem 1.6 (Algebraic thick subcategory theorem) *If $\mathcal{C}$ is a thick subcategory of $\mathcal{BP}_0$, then there exists an integer k such that $\mathcal{C} = \mathcal{BP}_k$.*

Let us conclude the introduction with some remarks.

- Theorem 3.4.2 of [Rav92] is the analog of Theorem 1.6 stated in a different category, $C\Gamma$, which is defined in terms of MU rather than BP.
- The BP-homology functor, $BP_*(\cdot) : \mathcal{CP}_0 \longrightarrow \mathcal{BP}_0$ sends the category $\mathcal{CP}_k$ into $\mathcal{BP}_k$. This comes from the fact [Rav84] that if $X \in \mathcal{CP}_0$ then

$$K(n)_*(X) = 0 \iff v_n^{-1}BP_*(X) = 0.$$

- Theorem 1.6 can be generalized to the abelian category of $P(n)_*(P(n))$-comodules, finitely presented over $P(n)_*$, which we denote by $\mathcal{P}(n)$. Similarly as for $\mathcal{BP}_0$ we can define the subcategories $\mathcal{P}(n)_k$ and prove the following.

Theorem 1.7 *If $\mathcal{C}$ is a thick subcategory of $\mathcal{P}(n)$, then there exists an integer $k \geq n$ such that $\mathcal{C} = \mathcal{P}(n)_k$.*

A further generalization of Theorem 1.6 can be obtained in the following setting. Let E_* be a commutative $P(n)_*$-algebra such that $E_* \otimes_{P(n)_*} -$ is an exact functor on $\mathcal{P}(n)$. In [Lan76] the second author gave sufficient conditions for exactness. (The necessity of these conditions was shown by Rudyak in [Rud86].) Define

$$E_*(E) = E_* \otimes_{P(n)_*} P(n)_*(P(n)) \otimes_{P(n)_*} E_*;$$

It can be made into a Hopf algebroid by extending the structure maps for $P(n)_*(P(n))$. Moreover $E_*(E)$ is a flat E_*-module because $P(n)_*(P(n))$ is a flat $P(n)_*$-module and if N is a E_*-module then

$$E_*(E) \otimes_{E_*} N \cong E_* \otimes_{P(n)_*} \left(P(n)_*(P(n)) \otimes_{P(n)_*} N\right).$$

If M is an object of $\mathcal{P}(n)$ then $E_* \otimes_{P(n)_*} M$ is an $E_*(E)$-comodule via the E_*-extension of the composite:

$$\begin{array}{lll} M \longrightarrow & P(n)_*(P(n)) \otimes_{P(n)_*} M & \\ & \downarrow & \\ & E_*(E) \otimes_{P(n)_*} M & \longrightarrow E_*(E) \otimes_{E_*} \left(E_* \otimes_{P(n)_*} M\right). \end{array}$$

Let $\mathcal{E}$ be the category whose objects are $E_* \otimes_{P(n)_*} M$ with $M \in \mathcal{P}(n)$ and morphisms $E_* \otimes f : E_* \otimes M_1 \longrightarrow E_* \otimes M_2$ with $f : M_1 \longrightarrow M_2$ in $\mathcal{P}(n)$; then $\mathcal{E}$ is an abelian category equipped with an exact functor:

$$E_* \otimes_{P(n)_*} - : \mathcal{P}(n) \longrightarrow \mathcal{E}.$$

The image of the subcategory $\mathcal{P}(n)_k$, written $\mathcal{E}_k$, satisfies:

$$\cdots \subset \mathcal{E}_{k+1} \subset \mathcal{E}_k \subset \cdots \subset \mathcal{E}_n = \mathcal{E}.$$

We are no longer claiming that the inclusions are strict. The thick subcategories of $\mathcal{E}$ can be described as follow:

Theorem 1.8 *If $\mathcal{C}$ is a thick subcategory of $\mathcal{E}$, then there exists an integer $k \geq n$ such that $\mathcal{C} = \mathcal{E}_k$.*

It should be emphasized that under the above assumption on E_*, the functor $E_* \otimes_{P(n)_*} P(n)_*(\cdot)$ is a homology theory [Lan76] taking its values in the category $\mathcal{E}$ as far as finite spectra are concerned.

2 The proof of Corollary 1.2

Let $D : \mathcal{CP}_0 \longrightarrow \mathcal{CP}_0$ be the anti-equivalence induced by the Spanier-Whitehead duality [Ada74]. If $X \in \mathcal{CP}_0$ and Y is any spectrum, the graded group $[X, Y]_*$ is isomorphic to $\pi_*(DX \wedge Y)$. We say that the maps $f : \Sigma^n X \longrightarrow Y$ and $\widehat{f} : S^n \longrightarrow DX \wedge Y$ are adjoint if they correspond to each other under the above isomorphism of groups. In particular the adjoint of the identity $X \longrightarrow X$ is a map $e : S^0 \longrightarrow DX \wedge X$. Recall that $X^{(i)}$ is a notation for the i-fold smash product $X \wedge \cdots \wedge X$.

Set $R = DW \wedge W$, a ring spectrum whose unit is e and whose multiplication is the composite

$$R \wedge R = DW \wedge W \wedge DW \wedge W \xrightarrow{DW \wedge De \wedge W} DW \wedge S^0 \wedge W = R.$$

The map $f: X \longrightarrow Y$ is adjoint to $\widehat{f}: S^0 \longrightarrow DX \wedge Y$ and $W \wedge f$ is adjoint to the composite

$$S^0 \xrightarrow{\widehat{f}} DX \wedge Y \xrightarrow{e \wedge DX \wedge Y} R \wedge DX \wedge Y,$$

which we denote by g. Set $F = R \wedge DX \wedge Y$. The map $W \wedge f^{(i)}$ is adjoint to the composite

$$S^0 \xrightarrow{g^{(i)}} F^{(i)} = R^{(i)} \wedge DX^{(i)} \wedge Y^{(i)} \longrightarrow R \wedge DX^{(i)} \wedge Y^{(i)},$$

the latter map being induced by the multiplication in R.

We want to show that $W \wedge f^{(k)}$ is null for large k; by adjointness it suffices to prove that $g^{(k)}$ is null for large k. The second statement of Theorem 1.1 implies that we only need to show that $BP \wedge g^{(i)}$ is null for large i, so we can take k to be an appropriate multiple of i. Let $T_i = R \wedge DX^{(i)} \wedge Y^{(i)}$ and let T be the direct limit of

$$S^0 \xrightarrow{g} T_1 \xrightarrow{T_1 \wedge \widehat{f}} T_2 \xrightarrow{T_2 \wedge \widehat{f}} T_3 \longrightarrow \cdots.$$

The desired conclusion will follow from showing that $BP \wedge T$ is contractible.

At this point we need to use the theory of Bousfield classes. Recall that the Bousfield class of a spectrum X (denoted $\langle X \rangle$) is the collection of spectra Z for which $X \wedge Z$ is not contractible. In [Rav84] it was shown that

$$\langle BP \rangle = \langle K(0) \rangle \vee \langle K(1) \rangle \vee \cdots \vee \langle K(n) \rangle \vee \langle P(n+1) \rangle.$$

By assumption, $K(n) \wedge T$ is contractible for all n. Therefore it suffices to show that $P(m) \wedge T$ is contractible for large m.

Since we are concerned only with finite spectra, we have for large enough m:

$$\begin{aligned} K(m)_*(W \wedge f) &= K(m)_* \otimes_{\mathbf{F}_p} H_*(W \wedge f; \mathbf{F}_p) \\ P(m)_*(W \wedge f) &= P(m)_* \otimes_{\mathbf{F}_p} H_*(W \wedge f; \mathbf{F}_p). \end{aligned}$$

Our hypothesis implies that both of these homomorphisms are trivial, so the smash product $P(m) \wedge T$ is contractible as required.

3 The proof of Theorem 1.6

The proof of Theorem 1.6 is a consequence of the filtration theorem of Landweber, namely

Theorem 3.1 *[Lan73] Each object $M \in \mathcal{BP}_0$ has a filtration*

$$0 = M_s \subset \cdots \subset M_1 \subset M_0 = M$$

in the category $\mathcal{BP}_0$, so that for $0 \leq i \leq s-1$ the quotient M_i/M_{i+1} is stably isomorphic to BP_*/I_{n_i} in $\mathcal{BP}_0$, where $I_{n_i} = (p, v_1, \cdots, v_{n_i-1})$ are invariant prime ideals of BP_*. (Stably isomorphic means isomorphic after a dimension shift.)

For $M \in \mathcal{BP}_0$ define $\mathrm{Spec}(M) = \{m \geq 1\colon\ v_{m-1}^{-1}M = 0\} \bigcup \{0\}$ (set as usual $v_0 = p$). If $M \neq 0$ then $\mathrm{Spec}(M)$ is a finite subset of $\mathbf{N}$ and is of the form:

$$\mathrm{Spec}(M) = \{0, 1, \cdots, N_M\}$$

with $N_M \geq 0$.

Let $\mathcal{C}$ be a thick subcategory of $\mathcal{BP}_0$. Define an integer k by:

$$\bigcap_{M \in \mathcal{C}} \mathrm{Spec}(M) = \{0, 1, \cdots, k\}.$$

From the definition of k, one has $\mathcal{C} \subset \mathcal{BP}_k$ and $\mathcal{C} \not\subset \mathcal{BP}_{k+1}$. Let M in $\mathcal{C}$ be such that

$$v_{k-1}^{-1}M = 0 \ \text{ and } \ v_k^{-1}M \neq 0,$$

and let

$$0 = M_s \subset \cdots \subset M_1 \subset M_0 = M$$

be a Landweber filtration of M. As $\mathcal{C}$ is thick and $M \in \mathcal{C}$, all the M_i's belong to $\mathcal{C}$ as well as all the quotients $M_i/M_{i+1} \cong BP_*/I_{n_i}$.

Localization being an exact functor, all the $v_{k-1}^{-1}M_i$ are null and hence $v_{k-1}^{-1}M_i/M_{i+1} \cong v_{k-1}^{-1}BP_*/I_{n_i} = 0$. Therefore

$$n_i \geq k \ \text{ for } \ 0 \leq i \leq s-1. \tag{3.2}$$

On the other hand, $v_k^{-1}M \neq 0$ implies the existence of a j for which $v_k^{-1}BP_*/I_{n_j} \neq 0$, which forces

$$n_j \leq k \ \text{ for some } j, \ \ 0 \leq j \leq s-1. \tag{3.3}$$

From (3.2) and (3.3) we obtain that $n_j = k$ for some j, $0 \leq j \leq s-1$, hence $BP_*/I_k \in \mathcal{C}$. Now it is fairly easy to prove by induction that $BP_*/I_{k+l} \in \mathcal{C}$ for all $l \geq 0$. Consider the exact sequence in $\mathcal{BP}_0$

$$0 \longrightarrow BP_*/I_{k+l} \xrightarrow{v_{k+l}} BP_*/I_{k+l} \longrightarrow BP_*/I_{k+l+1} \longrightarrow 0$$

where the first morphism is multiplication by v_{k+l}. The subcategory $\mathcal{C}$ being thick, $BP_*/I_{k+l} \in \mathcal{C}$ implies $BP_*/I_{k+l+1} \in \mathcal{C}$.

We are now ready to show the inclusion $\mathcal{BP}_k \subset \mathcal{C}$. Let N be an object in $\mathcal{BP}_k$ and $0 = N_s \subset \cdots \subset N_1 \subset N_0 = N$ be a Landweber filtration of N. We have seen that $v_{k-1}^{-1}N = 0$ implies $n_i \geq k$ for all $0 \leq i \leq s-1$ with, as usual, n_i such that $N_i/N_{i+1} \cong BP_*/I_{n_i}$. By downward induction on i we prove that $N_i \in \mathcal{C}$. This works as follows.

First $N_s = 0 \in \mathcal{C}$. Second, the short exact sequence in $\mathcal{BP}_0$

$$0 \longrightarrow N_{i+1} \longrightarrow N_i \longrightarrow BP_*/I_{n_i} \longrightarrow 0$$

is such that $N_{i+1} \in \mathcal{C}$ (by the inductive assumption) and $BP_*/I_{n_i} \in \mathcal{C}$ as $n_i \geq k$. From the thickness of $\mathcal{C}$ we obtain that $N_i \in \mathcal{C}$. For $i = 0$ we have $N \in \mathcal{C}$ and so $\mathcal{BP}_k = \mathcal{C}$, as required.

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