Lecture 1

A solution to the Arf-Kervaire invariant problem

Instituto Superior Técnico
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A poem by Frank Adams written in the 70s

The school at Northwestern is as fertile as manure, full of deep insights, some rather obscure. Mark loves those damn thetas like a sister or brother, and if you don't like one proof, he'll give you another.
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Background and history

Exotic spheres
The Pontrjagin-Thom construction
The J-homomorphism
The use of surgery
The Hirzebruch signature theorem
The Arf invariant
Browder's theorem

Spectral sequences
The Adams spectral sequence
The Adams-Novikov spectral sequence

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- It says that a certain algebraically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds $M$ is always zero.

The question answered by our theorem is nearly 50 years old. It is known as the Arf-Kervaire invariant problem. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved.
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The $\theta_j$ in the theorem is the name given to a hypothetical manifold or map between spheres for which the Arf-Kervaire invariant is nontrivial. It has long been known that such things can exist only in dimensions that are 2 less than a power of 2.
Our main result (continued)

\( \theta_1, \theta_2 \) and \( \theta_3 \) are the squares of the Hopf maps \( \eta \in \pi_1, \nu \in \pi_3 \) and \( \sigma \in \pi_7 \).
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Here is Hopf’s definition of the map η : S³ → S².
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Here is Hopf’s definition of the map \( \eta : S^3 \rightarrow S^2 \).

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- For $(z_1, z_2) \in S^3$, define

$$\eta(z_1, z_2) = \begin{cases} 
\frac{z_1}{z_2} & \text{for } z_2 \neq 0 \\
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\]
- **Maps** \( \nu : S^7 \rightarrow S^4 \) **and** \( \sigma : S^{15} \rightarrow S^8 \) **can be defined in a similar way using quaternions and Cayley numbers or octonions.**
Our main result (continued)

Main Theorem

The Arf-Kervaire elements \( \theta_j \in \pi_{2j+1-2}(S^0) \) do not exist for \( j \geq 7 \).

- \( \theta_4 \in \pi_{30} \) and \( \theta_5 \in \pi_{62} \) were constructed in the '60s and '70s. The latter is the subject of a paper by Barratt-Jones-Mahowald published in 1984.
- The status of \( \theta_6 \in \pi_{126} \) is still open.
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The work of Kervaire and Milnor

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**Milnor’s Theorem (1956)**

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Existence of nonsmoothable manifolds. *There is a 10-dimensional topological manifold with no differentiable structure.*
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**Milnor’s Theorem (1956)**

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Existence of nonsmoothable manifolds. *There is a 10-dimensional topological manifold with no differentiable structure.*

These theorems are opposite sides of the same coin.
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Such a $\Sigma^k$, when embedded in Euclidean space, has a (nonunique) framing on its normal bundle and thus represents an element in the framed cobordism group $\Omega_k^{\text{framed}}$. 
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Such a $\Sigma^k$, when embedded in Euclidean space, has a (nonunique) framing on its normal bundle and thus represents an element in the framed cobordism group $\Omega_k^{\text{framed}}$. By the Pontrjagin-Thom construction, $\Omega_k^{\text{framed}}$ is isomorphic to the stable $k$-stem $\pi_k(S^0)$.
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A $k$-dimensional framed manifold $M$ (such as $\Sigma^k$) can be embedded in a Euclidean space $\mathbb{R}^{n+k}$ in such a way that it has a tubular neighborhood $V$ homeomorphic to $M \times D^n$. 
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The resulting map $\tilde{f}_M : S^{n+k} \rightarrow S^n$ represents an element in the homotopy group $\pi_{n+k}(S^n)$, which for large $n$ is isomorphic to the stable $k$-stem $\pi_k$. 
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The resulting map $\tilde{f}_M : S^{n+k} \rightarrow S^n$ represents an element in the homotopy group $\pi_{n+k}(S^n)$, which for large $n$ is isomorphic to the stable $k$-stem $\pi_k$. Pontrjagin showed that a cobordism between $M_1$ and $M_2$ leads to a homotopy between $\tilde{f}_{M_1}$ and $\tilde{f}_{M_2}$. 
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The classification of exotic spheres (continued)

Two framings on a framed $k$-manifold $M^k \subset \mathbb{R}^{n+k}$ differ by a map $M^k \to SO(n)$, the special orthogonal group of $n \times n$ real matrices.
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It sends an exotic sphere $\Sigma^k$ to its framed cobordism class, modulo the indeterminacy related to the nonuniqueness of the framing. Kervaire-Milnor denote this homomorphism by $p'$ in their Lemma 4.5.

An element in the kernel of $\tau_k$ is represented by an exotic sphere $\Sigma^k$ bounding a framed manifold $M^{k+1}$. Milnor’s original $\Sigma^7$ was such an example, bounding a $D^4$-bundle over $S^4$, which can be framed.

An element in the cokernel of $\tau_k$ is a framed $k$-manifold which is not framed cobordant to a sphere.
We are studying the homomorphism

\[ \tau_k : \Theta_k \rightarrow \text{coker}_k J = \pi_k(S^0)/\text{im} J \]
The classification of exotic spheres (continued)

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\[ S^4 \xrightarrow{\Sigma \eta} S^3 \xrightarrow{\eta} S^2. \]

The preimage of a typical point in \( S^2 \) is an exotically framed torus \( S^1 \times S^1 \) in \( S^4 \).
The use of surgery

Let $M^n$ be a framed manifold, either closed or bounded by a sphere $\Sigma^{n-1}$. 

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When $n$ is odd, we can surger $M^n$ into a sphere $\Sigma^n$ or a disk $D^n$, whose boundary must be an ordinary sphere $S^{n-1}$.

This implies $\tau_k : \Theta_k \to \text{coker}_k J$ is onto when $k$ is odd and one-to-one when $k$ is even.
Obstructions in the middle dimension

When \( n = 2m \), we can surger our framed manifold \( M^{2m} \) into an \((m - 1)\)-connected manifold, but we may not be able to get rid of \( H^m(M) \).
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When \( m = 2\ell \) is even, the cup product gives us a pairing

\[
H^{2\ell}(M; \mathbb{Z}) \otimes H^{2\ell}(M; \mathbb{Z}) \to H^{4\ell}(M, \partial M; \mathbb{Z})
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represented by a symmetric unimodular matrix $B$ with even diagonal entries.

Such matrices have been classified over the real numbers up to the appropriate equivalence relation. The key invariant is the signature $\sigma(B)$, the difference between the number of positive and negative eigenvalues over $\mathbb{R}$, which is always divisible by 8.
An interesting matrix

Here is a symmetric matrix with even diagonal entries and signature 8.

\[ B = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \]
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2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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The Dynkin diagram for $E_8$

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\[ \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
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The nodes on the graph correspond to the rows/columns of the matrix.

Nodes $i$ and $j$ are connected by an edge iff $b_{i,j} \neq 0$. 
Implications of the Hirzebruch signature theorem

The Hirzebruch signature theorem relates the signature of a smooth oriented closed $4\ell$-manifold $M$ to its Pontrjagin numbers.

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If our $M^{4\ell}$ is bounded by a sphere diffeomorphic to $S^{4\ell-1}$, then we can close $M$ by attached a $4\ell$-ball. We get a new manifold $N^{4\ell}$ that is framed at every point except the center of that ball.
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Such a manifold is said to be *almost framed*. 
Hirzebruch’s formula implies that our signature $\sigma(N^{4\ell})$ is divisible by a certain integer related to the numerator of a Bernoulli number.
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On the other hand, there is a way to construct a framed $4\ell$-manifold bounded by a sphere $\Sigma^{4\ell-1}$ such that $\sigma(B)$ is any multiple of 8. This gives us 28 distinct differentiable structures on $S^7$. 
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The kernel of $\tau_{4\ell-1}$ is a large cyclic group whose order was determined by Kervaire-Milnor.
The one remaining case

To recap, we have a homomorphism

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To recap, we have a homomorphism

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We have not yet discussed the kernel for $k = 4\ell + 1$ or the cokernel for $k = 4\ell + 2$.

It turns out that the two groups are related. For each $\ell$, one is trivial iff the other is $\mathbb{Z}/2$. 
Framed 4ℓ + 2-manifolds

We have a framed 4ℓ + 2-manifold, possibly bounded by a sphere \( \Sigma^{4\ell+1} \). We can surger it into a 2\ell-connected manifold.
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There is a map (not a homomorphism) $\mu : H^{2\ell+1} \rightarrow \mathbb{Z}/2$ such that

$$\lambda(x, y) = \mu(x) + \mu(y) + \mu(x + y).$$
The Arf invariant

This map $\mu : H^{2\ell+1}(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ is either 1 most of the time or 0 most of the time.
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The Arf-Kervaire invariant $\Phi(M)$ of a framed $(4\ell + 2)$-manifold is defined to be the Arf invariant of its quadratic form.
The Arf-Kervaire invariant question

Is there a closed framed \((4\ell + 2)\)-manifold with nontrivial Arf-Kervaire invariant?
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If there is, then \(\tau_{4\ell+2}\) has a cokernel of order 2 and \(\tau_{4\ell+1}\) is one-to-one.

Kervaire answered the question in the negative for \(\ell = 2\). He constructed a framed 10-manifold bounded by an exotic 9-sphere. By coning off its boundary, he got his nonsmoothable closed topological 10-manifold.
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Enter stable homotopy theory

Algebraic topologists attacked this question vigorously in the 1960s. The best result was the following.

Browder's Theorem (1969)

A framed \((4\ell + 2)\)-manifold with nontrivial Arf-Kervaire invariant can exist only when \(\ell = 2j - 1\) for some integer \(j\). In that case it exists iff the Adams spectral sequence element \(h^{2j} \in E_2, 2j + 1\) is a permanent cycle.
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**Browder’s Theorem (1969)**

**Relation to the Adams spectral sequence.** A framed $(4\ell + 2)$-manifold with nontrivial Arf-Kervaire invariant can exist only when $\ell = 2^{j-1} - 1$ for some integer $j$. In that case it exists iff the Adams spectral sequence element

$$h^2_j \in E_2^{2, 2j+1} = \text{Ext}_A^{2, 2j+1}(\mathbb{Z}/2, \mathbb{Z}/2)$$

is a permanent cycle.
The classical Adams spectral sequence

Here $A$ denotes the mod 2 Steenrod algebra and

$$h_j \in E_2^{2,2j} = \text{Ext}_A^{1,2j}(\mathbb{Z}/2, \mathbb{Z}/2)$$

is the element corresponding to $Sq^{2j}$. 
The classical Adams spectral sequence

Here $A$ denotes the mod 2 Steenrod algebra and

$$h_j \in E_2^{2\cdot 2^j} = \text{Ext}^1_A(\mathbb{Z}/2, \mathbb{Z}/2)$$

is the element corresponding to $Sq^{2^j}$. It is defined for all integers $j \geq 0$. 
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Adams showed that $h_j$ is a permanent cycle only for $0 \leq j \leq 3$. 
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Adams showed that $h_j$ is a permanent cycle only for $0 \leq j \leq 3$. These $h_j$ represent $2\iota$ (twice the fundamental class) and the Hopf maps $\eta \in \pi_1$, $\nu \in \pi_3$ and $\sigma \in \pi_7$. 
The classical Adams spectral sequence

Here $A$ denotes the mod 2 Steenrod algebra and

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Adams showed that $h_j$ is a permanent cycle only for $0 \leq j \leq 3$. These $h_j$ represent $2\iota$ (twice the fundamental class) and the Hopf maps $\eta \in \pi_1$, $\nu \in \pi_3$ and $\sigma \in \pi_7$.

For $j \geq 4$ there is a nontrivial differential

$$d_2(h_j) = h_0 h_{j-1}^2.$$
The classical Adams spectral sequence (continued)

Here is a picture of the Adams spectral sequence for the prime 2 in low dimensions.
The classical Adams spectral sequence (continued)

Here is a picture of the Adams spectral sequence for the prime 2 in low dimensions.
Spectral sequences in stable homotopy theory

Spectral sequences have been used to study the stable homotopy groups of spheres for the past 50 years.
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Complex cobordism theory has proven to be extremely useful. The corresponding spectral sequence was first studied by Novikov in 1967 and is known as the *Adams-Novikov spectral sequence*. 
The Adams-Novikov spectral sequence for $p = 2$ in low dimensions

It is helpful to separate it into two parts having to do with $\nu_1$-periodic and $\nu_1$-torsion elements. These are related to the image and cokernel of $J$. 
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The box indicates a copy of $\mathbb{Z}_{(2)}$. 

\[ \begin{array}{cccccccccc}
0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\
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\[ \beta_{2/2} = h_2 = \theta_2 \quad \text{and} \quad \beta_{4/4} = h_3 = \theta_3. \]
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\[ \beta_{2/2} \quad \beta_2 \quad \beta_{4/4} \quad \beta_3 \quad \beta_{4/3} \quad \beta_{4/2} \quad \beta_4 \]
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\[
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Color coding is as before. **Blue lines** indicate multiplication by \( \nu = h_2 = \alpha_{2,2/2} \).
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Here is the $\nu_1$-torsion part.

\[
\begin{array}{cccccccccc}
\beta_{2/2} &=& h_2^2 &=& \theta_2 & & & & & \\
\beta_2 &=& & & & & \beta_{4/4} &=& h_3^2 &=& \theta_3 & \\
\beta_{3} &=& & & & & & & & \\
\beta_{4/3} &=& & & & & & & & \\
\beta_{4/2} &=& & & & & & & & \\
\beta_4 &=& & & & & & & &
\end{array}
\]

\[\beta_{2/2} = h_2^2 = \theta_2 \text{ and } \beta_{4/4} = h_3^2 = \theta_3.\]

Color coding is as before. Blue lines indicate multiplication by $\nu = h_2 = \alpha_{2,2}/2$. Green lines indicate group extensions.
The Adams-Novikov spectral sequence for $p = 2$ in low dimensions

Here is the $\nu_1$-torsion part.

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The first differential in this spectral sequence occurs in dimension 26.
The Arf-Kervaire invariant in the Adams-Novikov spectral sequence

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In the Adams-Novikov spectral sequence, is the element \( \theta_j = \beta_{2j-1}/2^{j-1} \in E_2^{2,2j+1} \) a permanent cycle?

It cannot be the target of a differential because its filtration is too low. We will show that it is the source of a nontrivial differential for \( j \geq 7 \).
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Our strategy

We will produce a map $S^0 \to M$, where $M$ is a nonconnective spectrum with the following properties.

(i) It has an Adams-Novikov spectral sequence in which the image of each $\theta^j$ is nontrivial.

(ii) It is 256-periodic, meaning $\Sigma^{256} \sim = M$.

(iii) $\pi_{-2}(M) = 0$.

(ii) and (iii) imply that $\pi_{254}(M) = 0$. If $\theta^7$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for $\theta^j$ for larger $j$ is similar, since $|\theta^j| = 2^{j+1} - 2 \equiv -2 \mod 256$ for $j \geq 7$. 
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- The $J$-homomorphism
- The use of surgery
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A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel
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