A homomorphism
\[ \varphi : MU_* \rightarrow R \]
is called an \emph{R-valued genus} and is equivalent by Quillen’s theorem that \( \varphi \) to a 1-dimensional formal group law over \( R \). It is also known that the functor
\[ X \mapsto MU_*(X) \otimes_\varphi R \]
is a homology theory if \( \varphi \) satisfies certain conditions spelled out in Landweber’s Exact Functor Theorem.

Now suppose \( E \) is an elliptic curve defined over \( R \). It is a 1-dimensional algebraic group, and choosing a local parameter at the identity leads to a formal group law \( \hat{E} \), the formal completion of \( E \). Thus we can apply the machinery above and get an \( R \)-valued genus.

For example, the \textit{Jacobi quartic}, defined by the equation
\[ y^2 = 1 - 2\delta x^2 + \epsilon x^4, \]
is an elliptic curve over the ring
\[ R = \mathbb{Z}[1/2, \delta, \epsilon]. \]
The resulting formal group law is the power series expansion of
\[ F(x, y) = \frac{x\sqrt{1 - 2\delta y^2 + \epsilon y^4} + y\sqrt{1 - 2\delta x^2 + \epsilon x^4}}{1 - \epsilon x^2 y^2}; \]
this calculation is originally due to Euler. The resulting genus is known to satisfy Landweber’s conditions, and this leads to one definition of elliptic cohomology.
THE HOPKINS-MAHOWALD AFFINE GROUP ACTION. The Weierstrass equation for a general elliptic curve is

\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]

Under the affine coordinate change

\[ y \mapsto y + r \quad \text{and} \quad x \mapsto x + sy + t \]

we get

\[
\begin{align*}
  a_6 & \mapsto a_6 + a_4 r + a_3 t + a_2 r^2 \\
        & \quad + a_1 r t + t^2 - t^3 \\
  a_4 & \mapsto a_4 + a_3 s + 2 a_2 r \\
        & \quad + a_1 (r s + t) + 2 s t - 3 r^2 \\
  a_3 & \mapsto a_3 + a_1 r + 2 t \\
  a_2 & \mapsto a_2 + a_1 s - 3 r + s^2 \\
  a_1 & \mapsto a_1 + 2 s.
\end{align*}
\]

This can be used to define an action of the affine group on the ring

\[ A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]. \]

Its cohomology is the \( E_2 \)-term of a spectral sequence converging to \( \pi_* (\text{tmf}) \).
Theorem 1. Let $C(p, f)$ be the Artin-Schreier curve over $\mathbb{F}_p$ defined by the affine equation
\[ y^e = x^p - x \quad \text{where } e = p^f - 1. \]
(Assume that $f > 1$ when $p = 2$.) Then its Jacobian has a 1-dimensional formal summand of height $(p - 1)f$.

Conjecture 2. Let $\tilde{C}(p, f)$ be the curve over $\mathbb{Z}_p[u_1, \ldots, u_{(p-1)f-1}]$ defined by
\[ y^e = x^p - x + \sum_{i=0}^{(p-1)f-2} u_{i+1} x^{p^{-1-[i/f]} y^{p^{i-1-p^{-1-[i/f]/f}}}. \]
Then its Jacobian has a formal 1-dimensional summand isomorphic to the Lubin-Tate lifting of the formal group law of height $(p - 1)f$.

Properties of $C(p, f)$:
- Its genus is $(p - 1)(d - 1)/2$.
- It has an action by the group $G = \mathbb{F}_p \rtimes \mu_{(p-1)e}$ given by
  \[ (x, y) \mapsto (\zeta^d x + a, \zeta y) \]
  for $a \in \mathbb{F}_p$ and $\zeta \in \mu_{(p-1)e}$. This group is a maximal finite subgroup of the $(p - 1)^{th}$ Morava stabilizer group, and it acts appropriately on the 1-dimensional formal summand.
- The case $f = 1$ was studied by Gorbunov-Mahowald.

Examples:
- $C(2, 2)$ and $C(3, 1)$ are elliptic curves whose formal group laws have height 2.
- $C(2, 3)$ has genus 3 and a 1-dimensional formal summand of height 3.
- $C(2, 4)$ and $C(3, 2)$ each has genus 7 and a 1-dimensional formal summand of height 4.
Theorem 3 (Honda). Let $A$ be a $\mathbb{Z}_p$-algebra with an automorphism $\sigma$ such that $a^\sigma$ is congruent to $a^p \mod p$. Then the strict isomorphism classes of $n$-dimensional formal group laws over $A$ correspond bijectively to the equivalence classes of matrices $H \in M_n(\mathbb{Z}_p)_{\sigma}\langle\langle T\rangle\rangle$ congruent to $pI_n$ modulo degree 1. $H$ and $f$ are related by the formula

$$f(x) = (H^{-1} \ast p)(x).$$

Examples:

- For $n = 1$ and $A = \mathbb{Z}_p$, let $H$ be the $1 \times 1$ matrix with entry $u = p - T^h$ for a positive integer $h$. Then

$$f(x) = \sum_{i \geq 0} \frac{x^{p^h}}{p^i}$$

and $F$ is the formal group law for the Morava K-theory $K(h)_*$.  

- Let $A = \mathbb{Z}_p[[u_1, u_2, \ldots u_{h-1}]]$ for a positive integer $h$, and let $u_i^p = u_i$. Let $H$ be the $1 \times 1$ matrix with entry

$$u = p - T^h - \sum_{0 < i < h} u_i T^i.$$

Then $f(x)$ is the logarithm for the Lubin-Tate lifting of the formal group law above.
Theorem 4 (Honda). For a curve $C$ of genus $g$, let
\[
\{\omega_1, \ldots, \omega_g\}
\]
be a basis for the space of holomorphic 1-forms of $C$ written as power series in a local parameter $y$, and let
\[
\psi_1 = \int_0^y \omega_1.
\]
If $H$ is a Honda matrix for the vector $(\psi_1, \ldots, \psi_g)$, then it is also one for $\hat{J}(C)$.

Theorem 5 (Tate). The determinant of the Honda matrix for a curve of genus $g$ is a polynomial of the form
\[
T^{2g} + \cdots + p^g.
\]
A basis for the holomorphic 1-forms for $C(p, f)$ is
\[
\{\omega_{i,j} : ei + pj < (e - 1)(p - 1) - 1\},
\]
where
\[
\omega_{i,j} = \frac{x^i y^j dx}{y^{e-1}}.
\]
We denote its integral of its expansion in terms of $y$ by $\psi_{ei + j + 1}$, which has power series expansion of the form
\[
y^{ei + j + 1} \sum_{n \geq 0} c_{ei + j + 1, n} y^n.
\]
Examples:

- For $C(2,3)$ (where $g=3$ and $m=7$) the integrals have the form
  $$\psi_1 \in y\mathbb{Q}[y^7]$$
  $$\psi_2 \in y^2\mathbb{Q}[y^7]$$
  $$\psi_3 \in y^3\mathbb{Q}[y^7]$$

  The orbits in $\mathbb{Z}/(7)$ under multiplication by 2 include
  $$\{1, 2, 4\} \quad \text{and} \quad \{3, 6, 5\}.$$

- For $C(3,2)$ (where $g=7$ and $m=16$) the integrals have the form
  $$\psi_1 \in y\mathbb{Q}[y^{16}]$$
  $$\psi_2 \in y^2\mathbb{Q}[y^{16}]$$
  $$\psi_3 \in y^3\mathbb{Q}[y^{16}]$$
  $$\psi_4 \in y^4\mathbb{Q}[y^{16}]$$
  $$\psi_5 \in y^5\mathbb{Q}[y^{16}]$$
  $$\psi_9 \in y^9\mathbb{Q}[y^{16}]$$
  $$\psi_{10} \in y^{10}\mathbb{Q}[y^{16}]$$

  The orbits in $\mathbb{Z}/(16)$ under the multiplication by 3 include
  $$\{1, 3, 9, 11\}, \{15, 13, 7, 5\}, \{2, 6\}, \{14, 10\}, \text{ and } \{4, 12\}.$$
References


