Higher chromatic generalizations of elliptic cohomology

Douglas C. Ravenel
University of Rochester
April 2, 2005

Happy Birthday, Martin and Don!

A 1-dimensional formal group law over and ring R leads to a homomorphism (called a genus)

$$\varphi : \pi_*(MU) \to R$$

by Quillen's theorem. The functor

$$X \mapsto MU_*(X) \otimes_\varphi R$$

is a homology theory if certain algebraic conditions on φ are satisfied; this is the Landweber Exact Functor Theorem.

Suppose E is an elliptic curve defined over R. It is a 1-dimensional algebraic group, and choosing a local paramater at the identity leads to a formal group law $\widehat{E}$, the formal completion of E. Thus we can apply the machinery above and get an R-valued genus.

For example the *Jacobi quartic*, defined by the equation

$$y^2 = 1 - 2\delta x^2 + \epsilon x^4,$$

is an elliptic curve over the ring

$$R = \mathbf{Z}[1/2, \delta, \epsilon].$$

The resulting formal group law is

$$F(x, y) = \frac{x\, R(y) + y\, R(x)}{1 - \epsilon x^2 y^2},$$

where

$$R(t) = \sqrt{1 - 2\delta t^2 + \epsilon t^4};$$

this calculation is originally due to Euler. The resulting genus is known to satisfy Landweber's conditions (Landweber-Ravenel-Stong), and this leads to one definition of elliptic cohomology theory.

A more general elliptic curve is defined by the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Under the affine coordinate change

$$y \mapsto y + r \qquad \text{and} \qquad x \mapsto x + sy + t$$

we get

$$\begin{aligned} a_6 &\mapsto a_6 + a_4\, r + a_3\, t + a_2\, r^2 \\ &\qquad +a_1\, r\, t + t^2 - r^3 \\ a_4 &\mapsto a_4 + a_3\, s + 2\, a_2\, r \\ &\qquad +a_1(r\, s + t) + 2\, s\, t - 3\, r^2 \\ a_3 &\mapsto a_3 + a_1\, r + 2\, t \\ a_2 &\mapsto a_2 + a_1\, s - 3\, r + s^2 \\ a_1 &\mapsto a_1 + 2\, s. \end{aligned}$$

This can be used to define a Hopf algrebroid $(A,\, \Gamma)$ with

$$\begin{aligned} A &= \mathbf{Z}[a_1, a_2, a_3, a_4, a_6] \\ \Gamma &= A[r, s, t] \end{aligned}$$

and right unit $\eta_R : A \to \Gamma$ given by the formulas above. It was first described by Hopkins and Mahowald in *From elliptic curves to homotopy theory.* Its Ext group is the E_2-term of a spectral sequence converging to $\pi_*(\mathrm{tmf})$. Tilman Bauer has written a nice account of this calculation.

It is known that the formal group law associated with an elliptic curve over a finite field can have height at most 2. Hence elliptic cohomology cannot give us any information about v_n-periodic phenomena for $n > 2$.

QUESTION: How can we find formal group laws of height > 2 attached to geometric objects (such as algebraic curves) and use them get insight into cohomology theories that go deeper into the chromatic tower?

PROGRAM:

- Let C be a curve of genus g over some ring R.
- Its Jacobian $J(C)$ is an abelian variety of dimension g.
- $J(C)$ has a formal completion $\widehat{J}(C)$ which is a g-dimensional formal group law.
- If $\widehat{J}(C)$ has a 1-dimensional summand, then Quillen's theorem gives us a genus associated with the curve C.

Theorem 1 (2002)**.** *Let $C(p,f)$ be the Artin-Schreier curve over $\mathbf{F}_p$ defined by the affine equation*

$$y^e = x^p - x \qquad \textit{where } e = p^f - 1.$$

(Assume that $(p,f) \neq (2,1)$.) Then its Jacobian $J(C(p,f))$ has a 1-dimensional formal summand of height $h = (p-1)f$.

Properties of $C(p,f)$:

- Its genus is $(p-1)(e-1)/2$, eg it is 0 in the excluded case, and 1 in the cases $(p,f) = (2,2)$ and $(3,1)$. In these cases C is an elliptic curve whose formal group law has height 2.
- Over $\mathbf{F}_{p^h}$ it has an action by the group

  $$G = \mathbf{F}_p \rtimes \mu_{(p-1)e}$$

  given by

  $$(x,y) \mapsto (\zeta^e x + a, \zeta y)$$

  for $a \in \mathbf{F}_p$ and $\zeta \in \mu_{(p-1)e}$.

  Let $\mathbf{G}_n$ denote the extension of the Morava stabilizer group S_n by the Galois group C_n. Given a finite subgroup $G \subset \mathbf{G}_n$, Hopkins-Miller can construct a "homotopy fixed point spectrum" E_n^{hG}. The group G above was shown by Hewett to be a maximal finite subgroup of $\mathbf{G}_n$ for $n = h = (p-1)f$. It acts on the 1-dimensional summand of $\widehat{J}(C(p,f))$ in the appropriate way.

Remarks

- This result was known to and cited by Manin in 1963. Most of what is needed for the proof can be found in Katz's 1979 Bombay Colloquium paper and in Koblitz' Hanoi notes.
- The original proof rests on the determination of the zeta function of the curve by Davenport-Hasse in 1934, and on some properties of Gauss sums proved by Stickelberger in 1890. The method leads to complete determination of $\widehat{J}(C(p,f))$.
- We have a simpler proof based on Honda's theory of commutative formal group laws developed in the early '70s. It does not rely on knowledge of the zeta function and is therefore more flexible. The starting point for it is the following consequence of the Lagrange inversion formula. If $z = x - x^p$, then x has the power series expansion

$$\begin{aligned} x &= \sum_{i\geq 0} \frac{1}{pi+1}\binom{pi+1}{i} z^{1+(p-1)i} \\ &= z + z^p + pz^{2p-1} + \frac{p(3p-1)}{2} z^{3p-2} + \cdots \end{aligned}$$

More remarks

- The curve above does not lead to a Landweber exact functor and cohomology theory. In order to get on we need to lift the curve to characteristic 0 in the right way. We will describe such a lifting below.
- Gorbunov-Mahowald studied this curve for $f = 1$. They found a lifting of the curve to characteristic zero associated with the Lubin-Tate lift of the formal group law of height $p-1$.

We want a lifting of $C(p,f)$ that admits a coordinate change similar to the one for the Weierstrass curve. The equation will have the form

$$y^e = x^p + \cdots$$

with (nonaffine) coordinate change

$$\begin{aligned} x &\mapsto x+\tilde{t} \qquad \text{where } \tilde{t} = \sum_{i=1}^{f} t_i y^{p^{f-1}-p^{i-1}} \\ y &\mapsto y \end{aligned}$$

The t_i above are related to the generators of the same name in $BP_*(BP)$.

In order to state this precisely we need some notation. Let

$$I = (i_1, \ldots, i_f)$$

be a set of nonnegative integers and define

$$\begin{aligned} |I| &= \sum_k i_k \\ ||I|| &= \sum_k (p^k - 1) i_k \\ t^I &= \prod_k t_k^{i_k} \\ I! &= \prod_k i_k! \end{aligned}$$

The coefficients in our equation will be formal variables a_I (where $a_0 = p!$) with topological dimension $2||I||$. Then the equation for our curve is

$$y^e \;=\; \sum_{i=0}^{p} \frac{x^{p-i}}{(p-i)!} \sum_{|I|=i} a_I y^{(ei-||I||)/p}$$

(recall that $e = p^f - 1$) and the effect of the coordinate change on the coefficients a_I is given by

$$a_I \mapsto \sum_{J+K=I} a_J \frac{t^K}{K!}.$$

For $f = 1$ the equation simplifies to the Gorbunov-Mahowald equation

$$y^{p-1} = x^p + \sum_{i=1}^{p} \frac{a_i x^{p-i}}{(p-i)!}$$

with coordinate change

$$a_i \mapsto a_i + \sum_{0<j<i} \frac{a_j t_1^{i-j}}{(i-j)!} + \frac{p! t_1^i}{i!}.$$

Theorem 2 (2004)**.** *The Jacobian of curve defined above over the ring*

$$A = \mathbf{Z}_p[a_I \colon 0 < |I| \le p]$$

has a 1-dimensional formal summand of height $(p-1)f$*.*

There is an associated Hopf algebroid

$$\Gamma = A[t_1, \ldots, t_f]$$

where each t_i is primitive and the right unit given by the coordinate change formula above. Note that

$$\eta_R(a_{p\Delta_i}) = a_{p\Delta_i} + \sum_{0<j<p} a_{j\Delta_i}\frac{t_i^{p-j}}{(p-j)!} + t_i^p.$$

This leads to a change-of-rings isomorphism

$$\mathrm{Ext}_\Gamma(A, A) = \mathrm{Ext}_{\Gamma'}(A', A')$$

where

$$\begin{aligned} A' &= A/(a_{p\Delta_1}, \ldots, a_{p\Delta_f}) \\ \text{and} \qquad \Gamma' &= A'[t_1, \ldots, t_f]/(\eta_R(a_{p\Delta_i}) - a_{p\Delta_i}). \end{aligned}$$

Note that Γ' is a free A'-module of rank p^f.

Conjecture 3. *For each (p, f) as above there is a spectrum generalizing* tmf *whose homotopy can be computed by an Adams-Novikov type spectral sequence with*

$$E_2 = \mathrm{Ext}_\Gamma(A, A).$$

University of Rochester, Rochester, NY 14627