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# THE ADAMS-NOVIKOV $E_2$ -TERM FOR A COMPLEX WITH $p$ CELLS

By DOUGLAS C. RAVENEL\*

We will describe the calculation of the  $p$ -component of the stable homotopy, in dimensions  $< p^3q$  where  $q = 2p - 2$ , of a certain finite complex  $X$  closely related to the sphere.  $X$  has  $p$  cells and

$$X = S^0 \cup e^q \cup \dots \cup e^{(p-1)q}$$

where each attaching map between adjacent cells is  $\alpha_1 \in \pi_{q-1}(S^0)$ .  $X$  is also the skeleton of a certain Thom spectrum  $T(1)$  of a bundle over  $\Omega S^{q+1}$  to be described below. The details of extracting  $\pi_*(S^0)$  from  $\pi_*(X)$  will be described elsewhere. Our description of the latter is simple enough to be intelligible and to convince the casual reader of its probable accuracy.

For  $p \geq 5$  our range of dimensions is new. Nakamura-Oka [4] has computed  $\pi_k(S^0)$  for  $k \leq (2p^2 + 4p + 1)q - 6$  and Aubry [2] has computed it for  $k \leq (3p^2 + 4p)q - 1$ .

We will compute the  $E_2$ -term of the Adams-Novikov spectral sequence (ANSS) for  $\pi_*(X)$ . It will follow for trivial reasons that there are no nontrivial differentials or group extensions in our range. We can describe our result briefly.  $E_2^{s,t}$  for  $s = 0, 1$  for  $X$  are closely related to the corresponding groups for  $S^0$  which are well known (see [3]).  $E_2^{s,t}$  is known for the sphere and is generated by the elements  $\beta_{i/j} \in E^{2,q(p+1)i-qj}$  where  $i > 0$  and  $j = 1$  unless  $p|i$  in which case  $1 \leq j \leq p$ . With the exception of  $\beta_1$  these elements all have nontrivial images in  $E_2^{s,t}$  for  $X$ .

To describe the rest of  $E_2$  we need some notation. Let  $P(1)$  be the Hopf algebra  $Z/(p)[t_1, t_2]/(t_1^p, t_2^p)$  with  $\dim t_i = 2(p^i - 1)$ ,  $t_1$  is primitive and  $t_2 = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2$ .  $P(1)$  is dual to the subalgebra of the  $\text{mod}(p)$  Steenrod algebra generated by  $P^1$  and  $P^p$ .

$$P(0) = Z/(p)[t_1]/(t_1^p) = H_*(X; Z/(p))$$

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is a  $P(1)$ -comodule. We compute  $\text{Ext}_{P(1)}^{s,t}(Z/(p), P(0))$  and denote it by  $R^{s,t}$ .  $R$  is a free module on  $2p - 1$  generators in degrees  $(2i, p^2qi)$  for  $0 \leq i \leq p - 1$  and  $(2i + 1, (p + ip + ip^2)q)$  for  $0 \leq i \leq p - 2$  over  $E(h_{20}) \otimes Z/(p)[b_{20}^p]$  where  $h_{20} \in R^{1,(p+1)q}$  and  $b_{20}^p \in R^{2p,(p^3+p^2)q}$ . Then our main result is

**THEOREM.** *The ANSS  $E_2$ -term  $E_2^{s,t}$  for  $X$  for  $s \geq 2, t < p^3q$  is  $A \oplus B \oplus C$  where  $A$  is the vector space spanned by  $\{\beta_{pi}, \beta_{pi+1} : i \geq 1\}$ ,  $B = R \otimes \{\gamma_k : k \geq 2\}$  where  $\gamma_k \in E_2^{3,kq(p^2+p+1)-(p+2)q}$ , and  $C^{s,t} = \otimes_{i \geq 0} R^{s+2i,t+i(p^2-1)q}$ . □*

This result is a reformulation of 4.13. For  $p = 5$  it is illustrated in Table 5.1.

In section 1 we will describe our method of calculation. In section 2 we compute the  $E_2$ -term for  $T(1)$ . In section 3 we construct a certain cochain complex and use it in section 4 to compute the  $E_2$ -term for  $X$ .

The work here was motivated in part by Aubry's calculation [2]. We commend him for his daring in breaking ground in this difficult area.

**1. The method of infinite descent and some homological algebra.**

The dual Steenrod algebra  $A_*$  is a commutative, noncocommutative Hopf algebra over  $Z/(p)$ . The usual definition of such an object is equivalent to the statement that it is a cogroup object in the category of commutative graded  $Z/(p)$ -algebras, i.e. given any such algebra  $R$  the set  $\text{Hom}(A_*, R)$  has a natural group structure induced by the coproduct on  $A_*$ .

Now  $BP_*(BP)$  in a graded commutative  $Z_{(p)}$ -algebra but it is *not* a cogroup object in the corresponding category because the coproduct

$$\Delta : BP_*(BP) \rightarrow BP_*(BP) \otimes_{BP_*} BP_*(BP)$$

is a map to the tensor product with respect to the  $BP_*$ -bimodule structure given by the right and left units  $\eta_R, \eta_L : BP_* \rightarrow BP_*(BP)$ . Consequently  $\text{Hom}(BP_*(BP), R)$  is not a group but a groupoid, and  $BP_*(BP)$  is not a Hopf algebra but a *Hopf algebroid*. Many of the notions of Hopf algebra theory, e.g. extensions and the Cartan-Eilenberg spectral sequence (CESS) carry over to Hopf algebroids, although the generalization is not always straightforward.

Let  $\Gamma(n) = BP_*(BP)/(t_1, t_2 \cdots t_{n-1})$ ; this is a quotient Hopf algebroid of  $BP_*(BP) = \Gamma(1)$ . Let  $A(n) = Z_{(p)}[v_1, v_2 \cdots v_n]$  and  $G(n) = A(n)[t_n]$ . Then  $G(n)$  is a sub-Hopf algebroid of  $\Gamma(n)$  and

$$(1.1) \quad G(n) \rightarrow \Gamma(n) \rightarrow \Gamma(n + 1)$$

is an extension of Hopf algebroids.

1.2. PROPOSITION

- (a)  $\text{Ext}_{\Gamma(1)}(BP_*, BP_*[t_1, t_2, \dots, t_{n-1}]) = \text{Ext}_{\Gamma(n)}(BP_*, BP_*)$
- (b)  $\text{Ext}_{\Gamma(n)}^0(BP_*, BP_*) = A(n - 1)$  and  $\text{Ext}_{\Gamma(n)}^{s,t}(BP_*, BP_*) = 0$  for  $s > 0$  when  $t < 2(p^n - 1)$ . □

(a) is a standard change-of-rings isomorphism (see 1.20) and (b) follows from the fact that  $A(n - 1)$  and  $\Gamma(n)$  are both isomorphic to  $BP_*$  (i.e.  $\Gamma(n)$  is trivial as a Hopf algebroid) in this range. Alternatively, using (a),  $BP_*[t_1, \dots, t_{n-1}]$  is isomorphic to  $BP_*$  ( $BP$ ) in this range, so the corresponding Ext group is simply  $BP_*$ .

Our method of infinite descent is to compute the Ext groups of (a) by downward induction on  $n$ , using (b) to start the process. There is a Cartan-Eilenberg spectral sequence (CESS) for the extension 1.1, but we prefer to use another spectral sequence which we will construct below. We do so by producing a double complex whose cohomology is  $\text{Ext}_{\Gamma(n)}(BP_*, M)$  for a left  $\Gamma(n)$ -comodule  $M$ . In the usual fashion this double complex yields two spectral sequences (both nontrivial) converging to the same thing; one is the CESS and the other is the one we prefer.

Before doing this we observe that each step in the induction has a topological interpretation in view of the following result.

1.3. Theorem. *For each  $n \geq 0$  there is a homotopy commutative ring spectrum  $T(n)$  with  $BP_*T(n) \cong BP_*[t_1, t_2 \cdots t_n]$  as a comodule over  $BP_*(BP)$ , so  $\text{Ext}_{\Gamma(n+1)}(BP_*, BP_*)$  is the  $E_2$ -term of the ANSS for  $\pi_*(T(n))$ .*

*Outline of proof.* By Bott periodicity  $BU = \Omega SU$ , so the inclusion  $SU(m) \rightarrow SU$  gives a vector bundle over  $\Omega SU(m)$ . An easy calculation shows  $H_*(\Omega SU(m)) = Z[b_1, b_2 \cdots b_{m-1}]$  where  $b_i \in H_{2i}(BU)$  is the standard generator. Let  $X(m)$  be the corresponding Thom spectrum. It is possible to obtain  $T(n)$  as a retract of the  $p$ -localization of  $X(p^n)$  by means of an idempotent map similar to Quillen's idempotent on  $MU$ . Quillen's method, as explained by Adams [1] depends on the existence of an orientation class  $x \in MU^2(\mathbb{C}P^\infty)$  having certain properties. A similar class can be found in  $X(m)^2(\mathbb{C}P^m)$ . Quillen's calculations can be mimicked through an appropriate range of dimensions to give the desired splitting.  $X(p^n)$  and

hence  $T(n)$  are homotopy commutative ring spectra since  $\Omega SU(m)$  is a double loop space. □

This result is not really relevant to the problem at hand since our calculation is purely algebraic and depends only on the (self-evident) existence of the comodule algebras  $BP_*[t_1, \dots, t_n]$ .  $T(1)$  can be constructed directly as a ring spectrum (not obviously commutative) as follows. Let  $S^q \rightarrow BU$  correspond to a generator of  $\pi_q(BU)$  and extend the map to  $\Omega S^{q+1}$ . The corresponding Thom spectrum is  $T(1)$ .

In our calculations we will use the following tool.

**1.4. PROPOSITION.** *Let  $0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow \dots$  be a long exact sequence (LES) of comodules over  $\Gamma(n)$ . Then there is a spectral sequence converging to  $\text{Ext}_{\Gamma(n)}^{i+j}(BP_*, M)$  with  $E_1^{i,j} = \text{Ext}_{\Gamma(n)}^i(BP_*, C^j)$  and  $d_r : E_r^{i,j} \rightarrow E_r^{i-r+1, j+r}$ . □*

We now construct the CESS and our substitute for it. Recall the standard method of defining Ext over a Hopf algebroid such as  $\Gamma(n)$ . A  $\Gamma(n)$ -comodule is *extended* if it has the form  $\Gamma(n) \otimes_{BP_*} M$  for some  $BP_*$ -module  $M$ . There is an acyclic complex  $D_{\Gamma(n)}M$  of extended comodules with  $D_{\Gamma(n)}^s M = \Gamma(n)^{\otimes s+1} \otimes M$  (with all tensor products over  $BP_*$ ) with  $d(\gamma_0 \otimes \dots \otimes \gamma_s \otimes m) = \sum_{0 \leq i \leq s} (-1)^i \gamma_0 \otimes \dots \otimes \Delta(\gamma_i) \otimes \dots \otimes \gamma_s \otimes m + (-1)^{s+1} \gamma_0 \otimes \dots \otimes \gamma_s \otimes \psi(m)$  for  $\gamma_i \in \Gamma(n)$  and  $m \in M$ . Using the inclusion  $M \rightarrow D^0M$  and the map  $D^0M \rightarrow M$  induced by the augmentation  $\epsilon : \Gamma(n) \rightarrow BP_*$  one sees that  $D_{\Gamma(n)}M$  is chain homotopy equivalent (CHE) to  $M$  (regarded trivially as a complex) as a complex of  $BP_*$ -modules. One also has  $\text{Ext}_{\Gamma(n)}^r(BP_*, D_{\Gamma(n)}^s M) = 0$  for  $r > 0$  and  $H^0(D_{\Gamma(n)}M) = M$  so 1.4 implies  $\text{Ext}_{\Gamma(n)}(BP_*, M)$  is the cohomology of the complex  $\text{Hom}_{\Gamma(n)}(BP_*, D_{\Gamma(n)}M)$  which we denote by  $\Omega_{\Gamma(n)}M$ . This is the *cobar complex* for  $M$ . Explicitly we have  $\Omega_{\Gamma(n)}^s M = \Gamma(n)^{\otimes s} \otimes M$  with  $d(\gamma_1 \otimes \dots \otimes \gamma_s \otimes m) = \sum_{i=1}^s (-1)^i \gamma_1 \otimes \dots \otimes \Delta(\gamma_i) \otimes \dots \otimes \gamma_s \otimes m + (-1)^{s+1} \gamma_1 \otimes \dots \otimes \gamma_s \otimes \psi(m)$ .

Now  $D_{\Gamma(n)}M$  is also an acyclic complex of extended  $\Gamma(n+1)$ -comodules (since  $\Gamma(n) = \Gamma(n+1) \otimes BP_*[t_n]$ ) so  $H(\text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M)) = \text{Ext}_{\Gamma(n+1)}(BP_*, M)$ . Moreover  $\text{Hom}_{\Gamma(n+1)}(BP_*, \Gamma(n)) = G(n)$  so  $\text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M)$  is a complex of  $G(n)$ -comodules. Furthermore an easy direct calculation shows

$$\begin{aligned} BP_* &= \text{Hom}_{G(n)}(A(n), \text{Hom}_{\Gamma(n+1)}(BP_*, \Gamma(n))) \\ &= \text{Hom}_{\Gamma(n)}(BP_*, \Gamma(n)), \end{aligned}$$

so  $\Omega_{\Gamma(n)}M = \text{Hom}_{G(n)}(A(n), \text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M))$ . The  $G(n)$ -complex  $\text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M)$  is CHE to the double complex  $D_{G(n)}\text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M)$ , so  $\Omega_{\Gamma(n)}M$  is CHE to the double complex

$$(1.5) \quad \text{Hom}_{G(n)}(A(n), D_{G(n)}\text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M)).$$

The resulting double complex yields two spectral sequences. If we take the cohomology of the inner complex first we get

$$E_1 = \text{Hom}_{G(n)}(A(n), D_{G(n)}(\text{Ext}_{\Gamma(n+1)}(BP_*, M)))$$

and

$$E_2 = \text{Ext}_{G(n)}(A(n), \text{Ext}_{\Gamma(n+1)}(BP_*, M));$$

this is the CESS.

1.6. THEOREM. *Let  $M$  be a left  $\Gamma(n)$ -comodule with the notation as in 1.1. There is a Cartan-Eilenberg spectral sequence (CESS) converging to  $\text{Ext}_{\Gamma(n)}(BP_*, M)$  with*

$$E_2^{i,j} = \text{Ext}_{G(n)}^i(A(n), \text{Ext}_{\Gamma(n+1)}^j(BP_*, M)) \quad \text{and} \quad d_r: E_r^{i,j} \rightarrow E_r^{i+r, j-r+1}.$$

*Proof.* Filtering by degree of the outer complex of (1.5) yields a spectral sequence with  $E_1 = \text{Ext}_{G(n)}(A(n), \text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M))$ . Now  $\text{Hom}_{\Gamma(n+1)}(BP_*, \Gamma(n) \otimes_{BP_*} M) = G(n) \otimes_{A(n)} M$  so  $\text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M)$  is a complex of extended  $G(n)$ -comodules. Hence the higher Ext groups vanish,  $E_1 = \text{Hom}_{G(n)}(A(n), \text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M))$ , and the spectral sequence collapses from this  $E_1$ . Moreover for any  $\Gamma(n)$ -comodule  $N$  we have  $\text{Hom}_{G(n)}(A(n), \text{Hom}_{\Gamma(n+1)}(BP_*, N)) = \text{Hom}_{\Gamma(n)}(BP_*, N)$ , so our  $E_1$ -term is

$$E_1 = \text{Hom}_{\Gamma(n)}(BP_*, D_{\Gamma(n)}M)$$

$$= \text{Ext}_{\Gamma(n)}(BP_*, M). \quad \square$$

Now suppose we have another complex  $C_n$  of  $G(n)$ -comodules and a map  $C_n \rightarrow \text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M)$  inducing an isomorphism in cohomology. Then we get a map from  $\Omega_{G(n)}C_n$  to the double complex of (1.5) inducing an isomorphism of  $E_1$ -terms in the first SS (i.e. an isomorphism

to the CESS  $E_1$ -term). It follows that we have an isomorphism of  $E_\infty$ -terms and that (using induction and the 5-lemma) the two total complexes have the same cohomology, namely  $\text{Ext}_{\Gamma(n)}(BP_*, M)$ . The second SS associated with  $\Omega_{G(n)}C_n$  converges to this Ext with

$$(1.7) \quad E_1^{i,j} = \text{Ext}_{G(n)}^i(A(n), C_n^j) \quad \text{and} \quad d_r : E_r^{i,j} \rightarrow E_r^{i-r+1,j+r}.$$

This is our substitute for the CESS. However the  $C_n$  we have in mind does not map to  $\text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M)$  but to another CHE complex which we now describe. First we need the *chromatic resolution*

$$(1.8) \quad 0 \rightarrow BP_* \rightarrow M^0 \rightarrow M^1 \rightarrow \dots$$

constructed inductively as follows.

Let  $N^0 = BP_*$  and define a short exact sequence (SES)

$$(1.9) \quad 0 \rightarrow N^n \rightarrow M^n \rightarrow N^{n+1} \rightarrow 0$$

by  $M^n = v_n^{-1}BP_* \otimes_{BP_*} N^n$  where  $v_0 = p$ .

Hence we have  $M^0 = BP_* \otimes \mathbf{Q}$  and  $N^1 = BP_* \otimes \mathbf{Q}/Z_{(p)}$ . This LES leads via 1.4 to the chromatic spectral sequence (CSS) which is studied in [3], but we are interested in it here for other reasons. Any  $\Gamma(n)$ -comodule  $M$  which is free as a  $BP_*$ -module can evidently be tensored with (1.8) and (1.9). Let  $CM$  denote the corresponding acyclic complex with  $H^0(CM) = M$ .

1.10. *Definition.* For a  $BP_*$ -free comodule  $M$  the total complexes associated with the double complexes  $D_{\Gamma(n)}CM$  and  $\Omega_{\Gamma(n)}CM$  are denoted by  $CD_{\Gamma(n)}M$  and  $C\Omega_{\Gamma(n)}M$ . The latter is the *chromatic cobar complex* of  $M$ .

1.11. **LEMMA.**  $CD_{\Gamma(n)}M$  and  $C\Omega_{\Gamma(n)}M$  are CHE to  $D_{\Gamma(n)}M$  and  $\Omega_{\Gamma(n)}M$ , so  $H^*(C\Omega_{\Gamma(n)}M) = \text{Ext}_{\Gamma(n)}(BP_*, M)$  for a  $BP_*$ -free comodule  $M$ .

*Proof.*  $CD_{\Gamma(n)}M$  is a resolution of  $M$  by extended  $\Gamma(n)$ -comodules in which all the maps are split as  $BP_*$ -module homomorphisms. Therefore standard arguments show it is CHE to  $D_{\Gamma(n)}M$  as a complex of  $\Gamma(n)$ -comodules, and the equivalence  $\Omega_{\Gamma(n)}M \rightarrow C\Omega_{\Gamma(n)}M$  follows.  $\square$

1.12. **THEOREM.** Let  $M$  be a  $BP_*$ -free  $\Gamma(n)$ -comodule and  $C_n$  a complex of  $G(n)$ -comodules admitting a map to  $\text{Hom}_{\Gamma(n+1)}(BP_*$ ,

$CD_{\Gamma(n)}M$  inducing an isomorphism in cohomology. Then there is a spectral sequence converging to  $\text{Ext}_{\Gamma(n)}(BP_*, M)$  with

$$E_1^{i,j} = \text{Ext}_{G(n)}^i(A(n), C_n^j) \quad \text{and} \quad d_r : E_r^{i,j} \rightarrow E_r^{i-r+1, j+r}.$$

*Proof.* In (1.7) we set up such a SS from a map  $C_n \rightarrow \text{Hom}_{\Gamma(n+1)}(BP_*, D_{\Gamma(n)}M)$  and by 1.11  $D_{\Gamma(n)}M$  is CHE to  $CD_{\Gamma(n)}M$ .  $\square$

1.13. *Remark.* In our examples  $C_n$  will be a subcomplex of the chromatic resolution  $CM$ , and the obvious map to  $\text{Hom}_{\Gamma(n+1)}(BP_*, CD_{\Gamma(n)}M)$  will not be mentioned.

$C\Omega_{\Gamma(n)}BP_*$  is of course bigger than  $\Omega_{\Gamma(n)}BP_*$ , but it is more convenient because many elements in  $\text{Ext}_{\Gamma(n)}(BP_*, BP_*)$  can be more easily represented as cocycles in  $C\Omega_{\Gamma(n)}BP_*$  than in  $\Omega_{\Gamma(n)}BP_*$ .

$N^n$  is generated as a  $Z_{(p)}$  module by fractions with whose numerators are monomials in  $BP_*$  and whose denominators are monomials in  $A(n - 1)$  with relations given by the condition that such an element (reduced to lowest terms) vanishes if its denominator is not divisible by the product  $p\nu_1 \cdots \nu_{n-1}$ . The CSS leads to a homomorphism

$$\eta : \text{Ext}_{\Gamma(k)}^0(BP_*, N^n) \rightarrow \text{Ext}_{\Gamma(k)}^n(BP_*, BP_*)$$

known as the Greek letter construction. Alternatively, primitives in  $N^n$  give cocycles in  $C\Omega_{\Gamma(k)}^n BP_*$  representing their images under  $\eta$ .

1.14. *Definition.* Let  $\alpha^{(n)}$  be the  $n$ th letter of the Greek alphabet. For  $t > 0$   $\alpha_t^{(n)} \in \text{Ext}^n$  is (up to sign) the image of

$$\frac{\nu_n^t}{p\nu_1 \cdots \nu_{n-1}} \in \text{Ext}^0(M^n)$$

under the map  $\eta$  above. (The sign is discussed in [3].)

These elements have been of interest for some time in view of the following result (see [3] for references).

1.15. **THEOREM.** For all  $t > 0$

(a) for  $p > 2$   $\alpha_t$  is a nontrivial permanent cycle in the ANSS corresponding to an element of order  $p$  in  $\pi_{qt-1}(S^0)$  which is in the image of the  $J$ -homomorphism;

(b) for  $p > 3 \beta_t$  is a nontrivial permanent cycle corresponding to an element of order  $p$  in  $\pi_{(p+1)qt-q-2}(S^0)$ ;

(c) for  $p > 5 \gamma_t$  is a nontrivial permanent cycle corresponding to an element of order  $p$  in  $\pi_{qt(p^2+p+1)-(p+2)q-3}(S^0)$ . □

Algebraically these elements are constructed using the SES's

$$0 \longrightarrow \Sigma^{\dim v_n} BP_* / I_n \xrightarrow{v_n} BP_* / I_n \longrightarrow BP_* / I_{n+1} \longrightarrow 0$$

where  $I_n = (p, v_1, \dots, v_{n-1}) \subset BP_*$  (this SES can be embedded in that of (1.9)), and the following result of Morava and Landweber.

1.16. THEOREM.  $\text{Ext}^0(BP_* / I_n) = Z/(p)[v_n]$  for each  $n > 0$ . For  $n = 0$ ,  $\text{Ext}(BP_*) = Z_{(p)}$ . □

Our strategy will be to use certain subcomplexes of the chromatic co-bar complex to compute  $\text{Ext}_{\Gamma(k)}(BP_*, BP_*)$ . We will need the following generalization of 1.16.

1.17. PROPOSITION.  $\text{Ext}^0_{\Gamma(k)}(BP_*, BP_*) = A(k - 1)$  and for  $n > 0$   $\text{Ext}^0_{\Gamma(k)}(BP_*, BP_* / I_n) = Z/(p)[v_n, v_{n+1}, \dots, v_{n+k-1}]$ . □

1.18. Definition. Let  $\Gamma$  be a Hopf algebroid over  $A$  (in the sense that  $BP_*(BP)$  is a Hopf algebroid over  $BP_*$ ). A  $\Gamma$ -comodule  $M$  is a weak injective if  $\text{Ext}^s_{\Gamma(n)}(A, M) = 0$  for  $s > 0$ . □

1.19. Definition. Let  $M$  and  $N$  be right and left  $\Gamma$ -comodules respectively. Then their cotensor product  $M \square_{\Gamma} N$  is the kernel of the map  $M \otimes_A N \rightarrow M \otimes_A \Gamma \otimes_A N$  sending  $m \otimes n$  to  $m' \otimes m'' \otimes n - m \otimes n' \otimes n''$ . □

The following change of rings isomorphism (of which 1.2(a) is a special case) will be useful in section 4.

1.20. THEOREM. Let  $\Gamma$  and  $\Sigma$  be Hopf algebroids over  $A$  and let  $f: \Gamma \rightarrow \Sigma$  be a Hopf algebroid homomorphism such that  $\Gamma$  is weak injective as a  $\Sigma$ -comodule and let  $N$  be a left  $\Sigma$ -comodule free over  $A$ . Then

$$\text{Ext}_{\Sigma}(A, N) = \text{Ext}_{\Gamma}(A, \Gamma \square_{\Sigma} N).$$

*Proof.* We will use the cotensor product instead of Hom, e.g. writing  $\text{Hom}_{\Sigma}(A, D_{\Sigma}N)$  as  $A \square_{\Sigma} D_{\Sigma}N$ . It suffices to show that the complexes whose cohomologies are the two Ext groups, i.e.  $A \square_{\Sigma} D_{\Gamma}N$  and  $A \square_{\Gamma} D_{\Gamma}$ .

$(\Gamma \square_{\Sigma} N)$ , are equivalent. The former is  $A \square_{\Gamma} \Gamma \square_{\Sigma} D_{\Sigma} N$  so it suffices to show  $\Gamma \square_{\Sigma} D_{\Sigma} N$  and  $D_{\Gamma}(\Gamma \square_{\Sigma} N)$  are CHE as complexes of  $\Gamma$ -comodules. The latter is an  $A$ -split resolution of  $\Gamma \square_{\Sigma} N$  by extended  $\Gamma$ -comodules, so it suffices to show the same is true of the former. Since  $\Gamma \square_{\Sigma} \Sigma = \Gamma$ ,  $\Gamma \square_{\Sigma} D_{\Sigma} N$  is an  $A$ -split complex of extended  $\Gamma$ -comodules so it remains only to show it is acyclic. But the weak  $\Sigma$ -injectivity of  $\Gamma$  implies that cotensoring it with an  $A$ -split exact sequence of  $\Sigma$ -comodules (such as  $D_{\Sigma} N$ ) free over  $A$  preserves exactness.  $\square$

**2. The calculation of  $\text{Ext}(BP_*[t_1])$ .** We assume throughout the rest of the paper that the prime  $p$  is odd and that we are in the range  $t < p^3q$ . From 1.2 we have

2.1. PROPOSITION.

$$\text{Ext}_{\Gamma(4)}^{s,t}(BP_*, BP_*) = \begin{cases} A(3) & \text{for } s = 0 \\ 0 & \text{for } s > 0. \end{cases} \quad \square$$

Note that  $A(3) \simeq BP_*$  in our range. Consider the SES

$$0 \rightarrow A(3) \rightarrow C_3^0 \rightarrow C_3^1 \rightarrow 0$$

where  $C_3^0 = A(3)[p^{-1}v_3]$  and  $C_3^1$  is the quotient. We claim that  $\text{Ext}_{\Gamma(3)}^s(BP_*, C_3^i) = 0$  for  $s > 0$  for both  $i = 0$  and  $1$ , so we can find  $\text{Ext}_{\Gamma(3)}(BP_*, BP)$  by studying the associated LES. To verify the claim and find  $\text{Ext}_{\Gamma(3)}^0$ , we find

$$C_3^0/(p) \simeq A(2)/(p)[t_3]$$

as a  $\Gamma(3)$ -comodule, giving

$$\text{Ext}_{G(3)}^s(A(3), C_3^0/(p)) = \begin{cases} A(2)/(p) & \text{for } s = 0 \\ 0 & \text{for } s > 0 \end{cases}$$

which gives

$$(2.2) \quad \text{Ext}_{G(3)}^s(A(3), C_3^0) = \begin{cases} A(2) & \text{for } s = 0 \\ 0 & \text{for } s > 0. \end{cases}$$

$C_3^1$  as an  $A(2)$ -module is generated by  $\left\{ \frac{v_3^i}{p^i} : 1 \leq i \leq p - 1 \right\}$

since  $v_3^p/p^p$  is out of our range. The  $G(3)$ -coaction is given by

$$\psi \left( \frac{v_3^i}{p^i} \right) = \sum_j \binom{i}{j} \frac{v_3^{i-j} \otimes t_3^j}{p^{i-j}}$$

and these binomial coefficients are all nonzero mod( $p$ ). It follows that each subquotient in the  $p$ -adic filtration of  $C_3^1$  is isomorphic to a suspension of  $A(2)/(p)[t_3]$  and that

(2.3)

$$\text{Ext}_{G(3)}^s(A(3), C_3^1) = \begin{cases} A(2)/(p) \otimes \left\{ \frac{v_3^i}{p} : 1 \leq i \leq p - 1 \right\} & \text{for } s = 0 \\ 0 & \text{for } s > 0 \end{cases}$$

Combining 2.2 and 2.3 and using 1.12 we get

2.4 LEMMA.

$$\text{Ext}_{\Gamma(3)}^s(BP_*, BP_*) = \begin{cases} A(2) & \text{for } s = 0 \\ A(2)/(p) \otimes \left\{ \frac{v_3^i}{p} : 1 \leq i \leq p - 1 \right\} & \text{for } s = 1 \\ 0 & \text{for } s \geq 2 \end{cases}$$

where we are identifying  $v_3^i/p \in \text{Ext}_{\Gamma(3)}^0(BP_*, C_3^1)$  with its image under  $\eta$  in  $\text{Ext}_{\Gamma(3)}^1(BP_*, BP_*)$ . □

To compute  $\text{Ext}_{\Gamma(2)}(BP_*, BP_*)$  we use the spectral sequence of 1.12. We will construct a suitable complex  $C_2$  of  $G(2)$ -comodules of the form  $C_2^0 \rightarrow C_2^1 \rightarrow C_2^2$  satisfying

- 2.5. (a)  $H^s(C_2) = \text{Ext}_{\Gamma(3)}^s(BP_*, BP_*)$ ,
- (b)  $C_2^i$  in a weak  $G(2)$ -injective (1.18) for  $i = 0$  or  $1$  and
- (c) the induced maps

$$\text{Ext}_{G(2)}^s(A(2), C_2^i) \rightarrow \text{Ext}_{G(2)}^s(A(2), C_2^{i+1})$$

are zero for  $i = 0$  and  $1$ .

Then by (1.12)  $\text{Ext}_{G(2)}(A(2), C_2)$  is the  $E_1$ -term of a SS converging to  $\text{Ext}_{\Gamma(2)}(BP_*, BP_*) = \text{Ext}(BP_*[t_1])$ . Moreover 2.5(b) and (c) imply that this SS collapses from  $E_1$  giving

$$(2.6) \quad \text{Ext}_{\Gamma(2)}^s(BP_*, BP_*) = \begin{cases} \text{Ext}_{G(2)}^0(A(2), C_2^s) & \text{for } s \leq 2 \\ \text{Ext}_{G(2)}^{s-2}(A(2), C_2^s) & \text{for } s \geq 2. \end{cases}$$

The obvious choice for  $C_2^0$  is  $A(2)[p^{-1}v_2]$ ; it satisfies 2.5(b) by an argument similar to that given for 2.2. 2.5(a) requires that the map  $C_2^0 \rightarrow C_2^1$  factor through the quotient  $B_2^1 = C_2^0/A(2)$ . However  $B_2^1$  does not satisfy 2.5(b);  $C_2^1$  will be a submodule of  $M^1$  containing  $B_2^1$ .  $B_2^1$  as an  $A(1)$ -module is generated by

$$\left\{ \frac{v_2^i}{p^i} : 1 \leq i \leq p^2 - p \right\}$$

and the  $G(2)$ -coaction is given by

$$\psi\left(\frac{v_2^i}{p^i}\right) = \sum_j \binom{i}{j} \frac{v_2^{i-j}}{p^{i-j}} \otimes t_2^j.$$

From this one can see that  $\text{Ext}_{G(2)}^0(A(2), B_2^1)$ , i.e. the primitives in  $B_2^1$  under this coaction, is generated by  $\{v_2^j/ip\}$ .

A  $C_2^1$  satisfying 2.5(b) with  $\text{Ext}_{G(2)}^0(A(2), B_2^1)$  must contain elements  $x_{i,j}$  such that the last term in the expansion of  $\psi(x_{i,j})$  is a unit multiple of  $v_2^j/ip \otimes t_2^j$ , for all  $i, j$  satisfying  $i > 0, j \geq 0$  and  $i + j \leq p^2 - p$ . The first instance where this fails in  $D_2^1$  is  $x_{1,p-1}$ ; the last term of  $\psi((v_2^p)/p^p)$  is  $v_2^2/p \otimes t_2^{p-2}$  rather than  $v_2/p \otimes t_2^{p-1}$ . If we divide by  $p$  we get  $\psi((v_2^p)/p^{1+p}) = \dots + v_2/p \otimes t_2^{p-1} + 1/p \otimes t_2^p$  and this last term is unacceptable because  $1/p$  is outside our  $\text{Ext}^0$ . However, by subtracting  $v_1^{-1}v_3/p$  (which is permitted since we are looking for a submodule of  $M^1$ , in which negative powers of  $v_1$  are allowed) we can eliminate the unwanted term in  $\psi(x_{1,p-1})$ , so we define

$$x_{1,p-1} = \frac{v_2^p}{p^{1+p}} - \frac{v_1^{-1}v_3}{p}.$$

More generally we define

$$x_{i,j-1} = \frac{v_2^{1+j-1}}{\binom{i+j-1}{i} ip^j} - \frac{v_2^{i-1}(v_1^{-1}v_3)^{j/p}}{j}$$

where the second term is nontrivial iff  $j \equiv 0 \pmod{p}$ . (A formula which appears to be valid in all dimensions is

$$x_{i,j} = \frac{v_2^{i+j}}{\binom{i+j}{j} ip^{j+1}} + \sum_{k \geq 1} \frac{(-1)^k p^{k-1} \binom{i-1}{k-1} v_2^{i-k} (v_1^{-1}v_3)^{(j+k)/p}}{j+k}.$$

Compare this situation with that of 3.8 below.)

We let  $C_2^1 \subset M^1$  be the sub- $A(2)$ -module generated by all the  $x_{i,j}$ . Then we have

$$(2.7) \quad \text{Ext}_{G(2)}^s(A(2), C_2^1) = \begin{cases} A(2) \otimes \left\{ \frac{v_2^i}{ip} : i > 0 \right\} & \text{for } s = 0 \\ 0 & \text{for } s > 0 \end{cases}$$

and the  $C_2^2$  which will satisfy 2.5(a) is the  $A(2)$ -module generated by  $\{(v_3^2/pv_1^i) : i > 0\}$ , i.e., we have

2.8 LEMMA. *A complex  $C_2$  of  $G(2)$ -comodules satisfying conditions 2.5 and 1.12 is the subcomplex of the chromatic resolution (1.4) given by*

$$C_2^0 = A(2)[p^{-1}v_2],$$

$$C_2^1 = A(2) \otimes \left\{ \frac{v_2^{i+j}}{\binom{i+j}{i} ip^{j+1}} - \frac{v_2^{i-1}(v_1^{-1}v_3)^{(j+1)/p}}{j+1} : i > 0, j \geq 0 \right\}$$

Where the second term vanishes unless  $j \equiv -1 \pmod{p}$ , and

$$C_2^2 = A(2) \otimes \left\{ \frac{v_3^i}{pv_1^i} \right\}. \quad \square$$

In order to complete the data necessary for (2.6) we need to compute  $\text{Ext}_{G(2)}(A(2), C_2^2)$ . We do this by filtering  $C_2^2$  by powers of  $v_1$ , and we find that each subquotient is a suspension of  $M = A(2)/(p, v_1)[t_2^p]$ . A routine calculation shows

$$\text{Ext}_{G(2)}(A(2), M) = Z/(p)[v_2] \otimes E(h_{20}) \otimes P(b_{20})$$

where

$$h_{20} = [t_2] \quad \text{and} \quad b_{20} = \sum_{0 < i < p} \frac{1}{p} \binom{p}{i} [t_2^i | t_2^{p-i}].$$

These elements can be identified up to sign with  $\hat{\alpha}_1 = v_2/p$  and  $\hat{\beta}_1 = v_3/pv_1$  respectively. Hence we get

2.9. THEOREM. For  $p > 2$  and  $t < p^3q$

$$\text{Ext}_{\Gamma(2)}^s(BP_*[t_1]) = \begin{cases} A(1) & \text{for } s = 0 \\ A(1) \otimes \left\{ \frac{v_2^i}{ip} : i > 0 \right\} & \text{for } s = 1 \\ \text{Ext}_{G(2)}^{s-2}(A(2), C_2^2) & \text{for } s \geq 2 \end{cases}$$

where

$$\text{Ext}_{G(2)}(A(2), C_2^2) = E(\hat{\alpha}_1) \otimes P(\hat{\beta}_1) \otimes \left\{ \frac{v_3^i v_2^j}{pv_1^i} : i > 0, j \geq 0 \right\}. \quad \square$$

By 1.2 and 1.3 this Ext is the ANSS  $E_2$ -term converging to  $\pi_*(T(1))$ . The spectral sequence collapses because there is nothing in filtration  $\geq 2p - 1$ , so we have

2.10. COROLLARY. For  $p > 2$  and  $k < p^3q - 3$ ,  $\pi_k(T(1)) = \bigoplus_s \text{Ext}_{\Gamma(2)}^{s, k+s}(BP_*, BP_*)$  as described in 2.9. □

A parallel calculation gives  $\text{Ext}_{BP_*}^{s,t}(BP)(BP_*, BP_*) = \text{Ext}^{s,t}$  for  $t < p^2q$ .

2.11. THEOREM. For  $p > 2$  and  $t < p^2q$

$$\text{Ext}_{BP_*}^{s,t}(BP)(BP_*, BP_*) = \begin{cases} Z_{(p)} \text{ in dimension } 0 & \text{for } s = 0 \\ Z_{(p)} \otimes \left\{ \frac{v_i^1}{ip} : i > 0 \right\} & \text{for } s = 1 \\ Z/(p)[\beta_1] \otimes E(\alpha_1) \otimes \{\beta_i : i > 0\} & \text{for } s \geq 2 \end{cases}$$

and for  $k < p^2q - 3$ ,  $\pi_k(S^0) = \bigoplus_s \text{Ext}^{s,s+k}$ .

3. The complex  $D_1$ . The restrictions of section 2, i.e.  $p > 2$  and  $t < p^3q$  are still in effect. We depart slightly from the methods of section 1; instead of analyzing the extension  $G(1) \rightarrow \Gamma(1) \rightarrow \Gamma(2)$  we look at  $H(2) \rightarrow \Gamma(1) \rightarrow \Gamma(3)$  where  $H(2) = A(2)[t_1, t_2]$  with the evident Hopf algebroid structure. *Until further notice we will abbreviate  $\text{Ext}_{H(2)}(A(2), M)$  by  $\text{Ext}(M)$  for an  $H(2)$ -comodule  $M$ .* By the methods of 1.12 we have

3.1. THEOREM. Let  $M$  be a  $BP_*$ -free  $\Gamma(1)$ -comodule (recall  $\Gamma(1) = BP_*(BP)$ ) and let  $D_1$  be a complex of  $H(2)$ -comodules admitting an  $H(2)$ -map to  $\text{Hom}_{\Gamma(3)}(BP_*, CD_{\Gamma(1)}M)$  inducing an isomorphism in cohomology. Then there is a spectral sequence converging to  $\text{Ext}_{\Gamma(1)}(BP_*, M)$  with

$$E_1^{i,j} = \text{Ext}^i(D_1^j) \quad \text{and} \quad d_r : E_r^{i,j} \rightarrow E_r^{i-r+1, j+r}. \quad \square$$

For the case  $M = BP_*$  we will construct a  $D_1$  satisfying the hypotheses above in addition to

- 3.2. (a)  $D_1$  is a subcomplex of the chromatic resolution  $CBP_*(1.8)$ ;
- (b) for  $i = 0, 1$   $D_1^i$  is a weak  $H(2)$ -injective (1.18) with  $\text{Ext}^0(D_1^i) = \text{Ext}_{\Gamma(1)}^i(BP_*, BP_*)$ ;
- (c)  $D_1^i = 0$  for  $i > 2$ .

The Ext groups of 3.2(b) are known, i.e.

3.3. THEOREM. (a) For  $p > 2$

$$\text{Ext}_{\Gamma(1)}^{0,t}(BP_*, BP_*) = \begin{cases} Z_{(p)} & \text{for } t = 0 \\ 0 & \text{for } t > 0 \end{cases}$$

and for all  $i > 0$ ,  $\text{Ext}_{\Gamma(1)}^{1,qi}(BP_*, BP_*)$  is the cyclic group generated by the cocycle  $v_1^i/ip \in C\Omega_{\Gamma(1)}BP_*$ , i.e. the order of this group is the largest power of  $p$  dividing  $pi$ .

(b) For  $p > 2$  and  $t > 0$ ,

$$\text{Ext}_{\Gamma(1)}^t(BP_*, M^1) = \begin{cases} \text{Ext}_{\Gamma(1)}^{1,t}(BP_*, BP_*) & \text{for } s = 0 \\ 0 & \text{for } s > 0 \end{cases}$$

where  $M^1$  is as in (1.8). □

References and a proof can be found in [3].

In constructing  $D_1$ , the obvious choice for  $D_1^0$  is  $A(2)[p^{-1}v_1, \overline{p^{-1}v_2}]$ ; filtering it by powers of  $p$  gives subquotients isomorphic to  $\overline{H(2)} = H(2) \otimes A(2)Z/(p)$  and we have  $\text{Ext}^0(D_1^0) = Z_{(p)}$  as required by 3.2(b).

$D_1^1$  is harder to get at and we will approach it rather indirectly. First we introduce a useful technical tool, which will be used in our proof that  $D_1^1$  exists (3.6).

**3.4. Definition.** Let  $G$  be a graded abelian  $p$ -group with  $G$  finite for all  $i$  and trivial except when  $q \mid i$ . Then the *Poincaré series* for  $G$  is  $\Sigma g_i x^i$  where the order of  $G^{iq}$  is  $p^{gi}$ .

The following characterization of weak injective comodules in terms of Poincaré series is useful.

**3.5. LEMMA.** *Let  $M$  be a connective torsion  $H(2)$ -comodule of finite type concentrated in dimensions divisible by  $q$ . Then the Poincaré series of  $M$  is dominated by that of  $H(2) \otimes \text{Ext}^0(M)$  with equality holding iff  $M$  is a weak injective. ( $H(2)$  could be replaced by any other Hopf algebroid considered in this paper.)*

*Proof.* We will construct a decreasing filtration  $\{F^i\}$  on  $M$  such that the associated bigraded comodule  $E_0M$  is annihilated by  $I = (p, v_1, v_2) \subset A(2)$  and  $\text{Ext}^0(E_0M) = E_0 \text{Ext}^0M$ , so  $\text{Ext}^0(E_0M)$  has the same Poincaré series as  $\text{Ext}^0(M)$ . Then we will prove the theorem by showing it for  $E_0M$ .

For any comodule  $M$  as above we will construct a subcomodule  $M' \subset M$  containing  $IM$  such that  $\text{Ext}^0(IM) = \text{Ext}^0(M')$  and the SES

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

induces a SES in  $\text{Ext}^0$ . Then the desired filtration can be defined by  $F^{k+1}M = (F^k M)'$ .

Define SES's

$$0 \rightarrow M'_i \rightarrow M \rightarrow M''_i \rightarrow 0$$

inductively as follows. Let  $M_0' = IM$  and let  $K_i = \text{Ext}^0(M''_i)/\text{im } \text{Ext}^0 M$ . Since  $M''_i$  is annihilated by  $I$  we can choose a splitting  $K_i \rightarrow \text{Ext}^0 M''_i$ .  $K_i$  is then a sub- $A(2)$ -module and therefore a subcomodule of  $M''_i$  and we can define  $M''_{i+1} = M''_i / K_i$ . Then we have SES's

$$\begin{array}{ccccccc} 0 & \longrightarrow & M'_i & \longrightarrow & M''_{i+1} & \longrightarrow & K_i \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_i & \longrightarrow & M & \longrightarrow & M_i \longrightarrow 0; \end{array}$$

$K_i$  was chosen so that  $K_i = \text{Ext}^0(K_i)$  maps monomorphically to  $\text{Ext}^1(M'_i)$ , so  $\text{Ext}^0(M'_i) = \text{Ext}^0(M''_{i+1})$ . It follows that  $\text{Ext}^0(IM) = \text{Ext}^0(M')$  where  $M' = \varinjlim M'_i$ .

Now we need to show that  $\text{Ext}^0(M)$  maps onto  $\text{Ext}^0(M'')$  where  $M'' = \varinjlim M''_i$ . This will follow from the fact that the connectivity of  $K_i$  increases with  $i$ . To see this suppose the first nontrivial element of  $K_i$  is in dimension  $n$ . Then  $M''_i$  has primitives there which do not pull back to primitives in  $M$ . But these elements are set equal to zero in  $M''_{i+1}$ , so  $K_{i+1}$  is  $n$ -connected.

Defining  $F^{k+1}M = (F^k M)'$  gives a decreasing filtration of  $M$  subordinate to the  $I$ -adic filtration (in the sense that  $E_0 M$  is annihilated by  $I$ ) with  $\text{Ext}^0(E_0 M) = E_0 \text{Ext}^0(M)$ . Hence it suffices to prove the lemma for  $E_0 M$ , in other words for comodules  $N$  annihilated by  $I$ .

Assume this  $N$  is  $(-1)$ -connected and let  $N^0$  be its 0-skeleton. We will argue by induction on dimension by constructing a SES of comodules

$$0 \rightarrow \tilde{N} \rightarrow N \rightarrow \bar{N} \rightarrow 0$$

such that the statement holds for  $\bar{N}$  by direct calculation, and  $\tilde{N}$  has higher connectivity than  $N$  and hence has the desired property through a higher range of dimensions by induction. The statement for  $N$  will follow since Poincaré series are additive with respect to extensions.

To get this SES, note that the  $A(2)$ -module splitting  $N \rightarrow N^0$  induces a comodule splitting  $N \otimes H(2) \rightarrow N^0 \otimes H(2)$ . Let  $\tilde{N}$  and  $\bar{N}$  be the kernel and image of the composite of this splitting with the comodule structure map  $\psi: N \rightarrow N \otimes H(2)$ . Now  $N^0 \subset \bar{N} \subset N^0 \otimes H(2)$  so  $\text{Ext}^0(\bar{N}) = N^0$ , which is a quotient of  $\text{Ext}^0(N)$ , so the statement clearly holds for  $\bar{N}$ . On the other hand  $\tilde{N}$  is 0-connected, so the result follows.  $\square$

3.6. LEMMA. For  $p > 2$  there exists an  $H(2)$ -comodule  $D_1^1$  satisfying 3.2.

*Proof.* Let  $\tilde{M}^1 = BP_* \square_{\Gamma(3)} M^1$ . From 1.6 and 3.3(b) we deduce that this  $H(2)$ -comodule has the same Ext groups in positive dimensions over  $H(2)$  as  $M^1$  has over  $BP_*(BP)$ . We will construct  $D_1^1$  as the direct limit of subcomodules  $K_i \subset \tilde{M}^1$ . Consider the following commutative diagrams in which rows and columns are SES's.

$$\begin{array}{ccccc}
 & & L_{i+1} & \equiv & L_{i+1} \\
 & & \uparrow & & \uparrow \\
 K_i & \longrightarrow & \tilde{M}^1 & \longrightarrow & L_i \\
 \parallel & & \uparrow & & \uparrow \\
 K_i & \longrightarrow & K_{i+1} & \longrightarrow & L'_i
 \end{array}$$

We define these comodules inductively by setting  $K_0 = D_1^0/A(2)$  and letting  $L'_i$  be the sub- $A(2)$ -module of  $L_i = \tilde{M}^1/K_i$  generated by the positive dimensional part of  $\text{Ext}^0(L_i)$ . It is a subcomodule of  $L_i$ ,  $K_{i+1}$  is the induced extension, and  $L_{i+1} = L_i/L'_i$ . Hence  $K_i, K_{i+1}$ , and  $L'_i$  are connective while  $L_i$  and  $L_{i+1}$  are not. From the LES for the right hand SES we deduce that the positive dimensional part of  $\text{Ext}^0(L_{i+1})$  is a subgroup of  $\text{Ext}^1(L'_i)$ , whose connectivity is necessarily greater than that of  $L'_i$ . It follows that the connectivity of  $L'_i$  increases with  $i$  so the limit  $K_\infty = \varinjlim K_i$  has finite type. The connectivity of the positive dimensional part of  $\text{Ext}^0(L_i)$  also increases with  $i$ . Consequently  $L_\infty = \varinjlim L_i$  has trivial  $\text{Ext}^0$  in positive dimensions. From the SES

$$0 \rightarrow K_\infty \rightarrow \tilde{M}^1 \rightarrow L_\infty \rightarrow 0$$

we deduce  $\text{Ext}^1(K_\infty) = 0$ . Now we will use 3.5 to conclude the higher Ext groups vanish as well and that  $K_\infty$  is therefore  $D_1^1$ . Consider the SES

$$0 \rightarrow K_\infty \rightarrow K_\infty \otimes_{A(2)} H(2) \rightarrow P \rightarrow 0$$

where  $P$  is the quotient. These comodules all satisfy the hypotheses of 3.5. This SES induces a SES of  $\text{Ext}^0$  groups by the above vanishing of  $\text{Ext}^1$ . The Poincaré series of the middle term above clearly satisfies the equality of 3.5, so the same is true of the two end terms and all three comodules are weak injective.  $\square$

We have constructed  $D_1^0$  and  $D_1^1$ . Since  $H^2(D_1) = \text{Ext}_{\Gamma(3)}^2(BP_*, BP_*) = 0$  in our range, we can take  $D_1^2$  to be the cokernel of the map  $D_1^0 \rightarrow D_1^1$ . First we compute its Poincaré series. The series for  $\bar{H}(2)$  and  $\text{Ext}_{\Gamma(1)}^1(BP_*, BP_*)$  are respectively  $(1/(1-x)(1-x^{1+p}))$  and  $\sum_{i \geq 0} (x^{p^i}/(1-x^{p^i}))$  (in our range only the first three terms of this sum are relevant). By 3.2 and 3.5 the series for  $D_1^1$  is the product of these two. To find those for the image we must subtract those for  $\text{im } D_1^0 = D_1^0/A(2)$  and for

$$H^1(D_1) = \text{Ext}_{\Gamma(3)}^1(BP_*, BP_*) \quad (\text{see 2.4}).$$

The latter group is a  $Z/(p)$ -vector space with basis  $\{(v_3^{1+i}v_2^jv_1^k/p) : i, j, k \geq 0\}$  so its Poincaré series is  $(x^{1+p+p^2}/(1-x)(1-x^{1+p})(1-x^{1+p+p^2}))$ .

The group  $D_1^0/A(2)$  is generated by  $\{(v_1^i v_2^j / p^m) : i + j \geq m > 0\}$ . In the subgroup generated by these elements with  $i \geq m$ , each generator can be written uniquely as  $(v_1/p)^{i+1} v_1^j v_1^k$  with  $i, j, k \geq 0$ , so the series for this subgroup is  $(x/(1-x)^2(1-x^{1+p}))$ . Each of the remaining generators can be uniquely written as  $(v_1/p)^i v_2^j (v_2/p)^{1+k}$  with  $i, j, k \geq 0$  so the series for the quotient is  $(x^{1+p}/(1-x)(1-x^{1+p})^2)$ . Combining these results gives

**3.7. PROPOSITION.** *If  $D_1$  is a complex as in 3.2, then the Poincaré series for the image of  $D_1^1$  in  $D_1^2$  is*

$$\frac{1}{(1-x^{1+p})} \left[ \frac{x^p}{(1-x^p)(1-x^{p+1})} + \frac{x^{p^2}(1-x^p)}{(1-x)(1-x^{p^2})(1-x^{p^2+p})} + \frac{x^{p^2+p}}{(1-x^{p^2+p})(1-x^{p^2+p+1})} \right] \quad \square$$

Our reason for writing the series in this form will become apparent presently. Note that the third term is the series for  $C_2^2$  (2.8).

Next observe that 3.2 implies that  $D_1$  satisfies 2.5(a) and (b) (but not (c)). Hence there is a spectral sequence converging to  $\text{Ext}_{\Gamma(2)}(BP_*, BP_*)$  collapsing from  $E_2$  (rather than  $E_1$ , since 2.5(c) is not satisfied) with  $E_1^{i,j} = \text{Ext}_{G(2)}^i(A(2), D_1^j)$  and  $d_1 : E_1^{i,j} \rightarrow E_1^{i,j+1}$ .

From 2.9 we know this  $E_2$ -term and we can use it to learn more about  $D_1$ . One sees easily that

$$\text{Ext}_{G(2)}^0(A(2), D_1^0) = A(1)[p^{-1}v_1].$$

Since  $E_2^{0,0} = A(1)$  we find that  $\text{im } d_1 \subset E_1^{0,1}$  is the  $A(1)$ -module generated by  $\{(v_1^i/p^i) : i > 0\}$ . This is not a weak injective over  $G(1)$  as we can see by applying the  $G(1)$  analogue of 3.5; the Poincaré series here is  $x/(1-x)^2$ . On the other  $E_1^{0,1} = \text{Ext}_{G(2)}^0(A(2), D_1^1)$  must be one, since

$$\text{Ext}_{H(2)}(A(2), D_1^1) = \text{Ext}_{G(1)}(A(1), E_1^{0,1}) \quad \text{by 1.6.}$$

**3.8. LEMMA.** *Let  $C_1^1 \subset M^1$  (1.8) be the  $A(1)$ -submodule generated by  $v_1^i/p^i$  and  $v_1^i/p^{i+1} - v_1^{-i}v_2^i/p$  for  $i > 0$ . Then  $C_1^1$  is a weak injective over  $G(1)$  with*

$$\text{Ext}_{G(1)}^0(A(1), C_1^1) = \text{Ext}_{\Gamma(1)}^1(BP_*, BP_*, BP_*).$$

*Proof.* We have to show that  $C_1^1$  is indeed a  $G(1)$ -comodule with the appropriate  $\text{Ext}^0$ . Then the result will follow from 3.5 once we have computed its Poincaré series. Consider the  $G(1)$ -comodule  $\tilde{M}^1 = BP_* \square_{\Gamma(2)} M^1$  where  $M^1$  is the chromatic comodule of (1.8). Each element of  $C_1^1$  is primitive over  $\Gamma(2)$  so  $C_1^1$  is an  $A(1)$ -submodule of  $\tilde{M}^1$ .

To see that it is a subcomodule let  $\tilde{C}_1^1 = \text{im } d_1 \subset C_1^1$  be the  $A(1)$ -submodule generated by  $v_1^i/p^i$  for  $i > 0$ , and let  $\bar{C}_1^1 = C_1^1/\tilde{C}_1^1$ . Then  $\bar{C}_1^1$  is clearly a comodule and

$$\psi \left( \frac{v_1^{ip}}{ip^{ip+1}} - \frac{v_1^{-i}v_2^i}{ip} \right) \equiv \sum_{0 \leq j \leq i} \frac{\binom{i}{j}^j}{i} \left[ \frac{v_1^{jp}}{jp^{jp+1}} - \frac{v_1^{-i}v_2^j}{jp} \right] \otimes t_1^{(i-j)p}$$

modulo  $\tilde{C}_1^1$  so  $C_1^1$  is a subcomodule of  $\tilde{M}^1$ . Now

$$\text{Ext}_{G(1)}^0(A(1), C_1^1) \subset \text{Ext}_{G(1)}^0(A(1), \tilde{M}^1) = \text{Ext}_{\Gamma(1)}^0(BP_*, M^1).$$

This group is known by 3.3 to be isomorphic to  $\text{Ext}_{\Gamma(1)}^1(BP_*, BP_*)$  in positive dimensions. Since every element of this group is in  $C_1^1$  (3.3) we have the desired isomorphism.

The Poincaré series for  $\tilde{C}_1^1$  is  $x/(1-x)^2$  while that of  $\bar{C}_1^1$  is readily seen to be

$$\frac{x^p}{(1-x)(1-x^p)} + \frac{x^{p^2}}{(1-x)(1-x^{p^2})}$$

so  $C_1^1$  has the series required by 3.5. □

Now the SS we are studying, i.e., that converging to  $\text{Ext}_{\Gamma(2)}(BP_*, BP_*)$  with  $E_1 = \text{Ext}_{G(2)}(A(2), D_1^1)$  must have  $E_2^{0,1} = \text{Ext}_{G(2)}^1(BP_*, BP_*)$ , the  $A(1)$ -module generated by  $v_2^i/ip$  for  $i > 0$ . We can conclude

**3.9. PROPOSITION.** *If  $D_1$  is as above then the image of  $\text{Ext}_{G(2)}^0(A(2), D_1^1)$  in  $\text{Ext}_{G(2)}^0(A(2), D_1^2)$  is the  $A(1)$ -module generated by  $v_2^i/ipv_1^i$  for  $i > 0$ . Moreover, there is an exact sequence  $\text{Ext}_{G(2)}^0(A(2), D_1^1) \rightarrow \text{Ext}_{G(2)}^0(A(2), D_1^2) \rightarrow \text{Ext}_{G(2)}^0(A(2), C_2^2) \rightarrow 0$  and an isomorphism  $\text{Ext}_{G(2)}^s(A(2), D_1^2) \simeq \text{Ext}_{G(2)}^s(A(2), C_2^2)$  for all  $s > 0$ , where  $C_2^2$  is as in 2.8. □*

We wish to compare this result with 3.7 by computing the Poincaré series for this group. The subgroup of exponent  $p$  is generated by  $\{(v_2^{i+1+j}/pv_1^j) : i, j \geq 0\}$  so its series is  $(x^p/(1-x^p)(1-x^{1+p}))$ . The elements of order  $p^2$  (there are no elements of order  $p^3$  in our range) are generated by

$$\left\{ \frac{v_2^{p(i+1+j)} v_1^k}{p^2 v_1^{pi}} : i, j, k \geq 0, k < p \right\}$$

so the corresponding series is  $(x^{p^2}(1-x^p)/(1-x^{p^2})(1-x^{p^2+p})(1-x))$ .

Multiplying these two series by  $(1/(1-x^{1+p}))$  gives the first two terms in 3.7.

**4. The calculation of  $\text{Ext}(BP_*(X))$ .** In the last section we constructed a complex  $D_1$  of  $H(2)$ -comodules suitable for the SS of 3.1. Moreover 3.2(b) implies that  $\text{Ext}_{H(2)}^s(A(2), D_1^2) = \text{Ext}_{\Gamma(1)}^{s+2}(BP_*, BP_*)$  in our range. (We are still abbreviating  $\text{Ext}_{H(2)}(A(2), M)$  by  $\text{Ext}(M)$ ). To get at this group we could use the CESS (1.6) for the extension  $G(1) \rightarrow H(2) \rightarrow G(2)$ , i.e.

4.1. PROPOSITION. *There is a CESS (1.6) converging to  $\text{Ext}_{H(2)}(A(2), D_1^2)$  with  $E_2^{s,t} = \text{Ext}_{G(1)}^s(A(1), \text{Ext}_{G(2)}^t(A(2), D_1^2))$ .  $\square$*

The group  $\text{Ext}_{G(2)}(A(2), D_1^2)$ , which we denote by  $E$  is described in 3.9, which says there is a SES of bigraded  $G(1)$ -comodules

$$(4.2) \quad 0 \rightarrow B \rightarrow E \rightarrow F \rightarrow 0,$$

where  $B = \text{im } \text{Ext}_{G(2)}(A(2), D_1^2)$  is the  $A(1)$ -module generated by  $\{v_2^i / ipv_1^i : i > 0\}$  concentrated in degree 0 (note that for  $i \geq p^2$  these elements are out of our range) and  $F = \text{Ext}_{G(2)}(A(2), C_2^2)$  as given by 2.9, which says  $F^0 = \text{Ext}_{G(2)}^0(A(2), C_2^2)$  is the  $Z/(p)$ -vector space generated by  $\{v_2^i v_1^j / pv_i : i \geq 0, j > 0\}$  and

$$4.3. \quad F = F^0 \otimes E(h_{20}) \otimes P(b_{20}0).$$

4.4. Definition. For a right  $BP_*(BP)$ -comodule  $M$  let  $r_i : M \rightarrow \Sigma^i M$  be the group homomorphism defined by  $\psi(m) = \Sigma r_i(m) \otimes t_1^i \dots$ , where the other terms involve  $t_k$  for  $k > 1$ . These operations are similarly defined for  $G(1)$ - and  $H(2)$ -comodules. Let  $X^n \subset G(1)$ ,  $Y^n \subset H(2)$  and  $T^n \subset BP_*(BP)$  denote the submodule generated by  $\{t_i^i : 0 \leq i \leq n\}$ . A  $G(1)$ -comodule  $M$  is  $k$ -free if the comodule tensor product  $M \otimes X^{p^k-1}$  is a weak  $G(1)$  injective.

Our ultimate goal in this paper is to compute  $\text{Ext}_{BP_*(BP)}(BP_*, T^{p-1})$  for which we can use the SS associated with the complex  $D_1 \otimes Y^{p-1}$ . The main difficulty here is computing  $\text{Ext}_{H(2)}(A(2), D_1^2 \otimes Y^{p-1})$ , which we do in two stages. First we compute  $\text{Ext}_{H(2)}(A(2), D_1^2 \otimes Y^{p^2-1})$  using a CESS with  $E_2$ -term  $\text{Ext}_{G(1)}(A(1), E \otimes X^{p^2-1})$ , then we use another SS (to be described below) to get the Ext group for  $D_1^2 \otimes Y^{p-1}$  from that for  $D_1^2 \otimes Y^{p^2-1}$ .

To compute  $\text{Ext}_{G(1)}(A(1), X^{p^2-1} \otimes E)$  we will construct a SES of bigraded  $G(1)$ -comodules

$$(4.5) \quad 0 \rightarrow E \rightarrow \tilde{E} \rightarrow \bar{E} \rightarrow 0$$

with  $\tilde{E}$  and  $\bar{E}$  2-free (4.4). Hence the above Ext group will vanish in all degrees 1 and the CESS will collapse.

4.6. LEMMA. *Let  $M$  be a  $G(1)$ -comodule and let  $L \subset M$  be the subcomodule  $\bigcap_{i > p^2} \ker r_i$ . Then  $\text{Ext}_{G(1)}^0(A(1), M \otimes X^{p^2-1})$  is isomorphic to  $L$  as a group.*

*Proof.* Let  $m \in L$ . Then one easily sees that  $\sum_{0 < i < p^2} r_i(m) \otimes t_1^i \in M \otimes X^{p^2-1}$  is primitive. Now we show that any primitive  $x \in M \otimes X^{p^2-1}$  must have this form. Write  $x = \sum_{0 \leq i \leq p^k} x_i \otimes t_1^i$  and assume inductively that  $x_i = (-1)^i r_i(x_0)$  for  $i < m$ . Then we have

$$\begin{aligned} 0 &= r_m(x) = \sum_{i,j \geq 0} r_j(x_i) \otimes r_{m-j}(t_1^i) \\ &= \sum r_j(x_i) \otimes \binom{i}{m-j} t_1^{i-j+m}. \end{aligned}$$

Collecting terms where the exponent of  $t_1$  is zero gives

$$\begin{aligned} 0 &= \sum r_j(x_{m-j}) = x_m + \sum_{j>0} r_j(x_{m-j}) \\ &= x_m + \sum_{j>0} (-1)^j r_j(r_{m-j}(x_0)) \\ &= x_m + \sum_{j>0} (-1)^j \binom{m}{j} r_m(x_0) \end{aligned}$$

which gives  $x_m = (-1)^i r_m(x_0)$ . □

Now we will construct the SES (4.5). We obtain  $\tilde{E}$  from  $E$  by adjoining  $\{v_2^{-i} v_3^j / p v_i : 0 < i < j\} \otimes E(h_{20}) \otimes P(b_{20})$  so we have

$$\bar{E} = \{v_3^j / p v_1 v_2^i : 0 < i < j\} \otimes E(h_{20}) \otimes P(b_{20}).$$

Now we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & F \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & \tilde{E} & \longrightarrow & \tilde{F} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \bar{E} & = & \bar{E} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

where  $\tilde{F}$  is the evident extension of  $F$  by  $\bar{E}$ . We will show  $B, \tilde{F}, \bar{E}$  and hence  $\tilde{E}$  are 2-free using 4.6 and the  $G(1)$  form of 3.5

For  $B$  we have

$$\sum_{j \geq 0} r_j \left( \frac{v_2^j}{ipv_1^j} \right) = \frac{(v_2 + v_1 t_1^p - v_1^p t_1)^i}{ipv_1^i}$$

so by 4.6  $\text{Ext}_{G(1)}^0(A(1), B \otimes X^{p^2-1})$  is isomorphic to the kernel of multiplication by  $v_1^p$ . To compute the Poincaré series for this group, note that the subgroup of exponent  $p$  is generated by  $\{v_2^{i+j}/pv_1^j : 1 \leq j \leq p, i \geq 0\}$  which has series  $(x^p/(1 - x^{p+1}))(1 - x^{p^2})/(1 - x^p)$ ; the elements of order  $p^2$  are spanned by  $\{v_2^{ip}/p_2v_1^j : 1 \leq j \leq p, i \geq 1\}$  so the corresponding series is  $(x^{p^2}(1 - x^p)/(1 - x)(1 - x^{p+p^2}))$ . Hence,  $B \otimes X^{p^2-1}$  will be a weak injective by 3.5 if we can show that its series is  $(1 - x)^{-1}$  times the sum of the two above. But from 3.7 we know the series for  $B$  is

$$\frac{x^p}{(1 - x^p)(1 - x^{1+p})} + \frac{x^{p^2}(1 - x^p)}{(1 - x)(1 - x^{p^2})(1 - x^{p+p^2})}$$

while that for  $X^{p^2-1}$  is  $(1 - x^{p^2}/1 - x)$  so the result follows.

Next we deal with  $\tilde{F} = \tilde{F}^0 \otimes E(h_{20}) \otimes P(b_{20})$  where  $\tilde{F}^0$  is the vector space spanned by  $\{v_3^{i+1}v_2^{-i+j}/pv_1^j : i, j \geq 0\}$ . Its series is

$$x^{p+p^2}/(1 - x^{1+p})(1 - x^{p^2}).$$

Now we claim

$$\text{Ext}_{G(1)}^0(A(1), \tilde{F}^0 \otimes X^{p^2-1}) = \text{Ext}_{G(1)}^0(A(1), \tilde{F}^0).$$

The latter is spanned by  $\{v_3 v_2^j / pv_1^j : j > 0\}$  so its series is

$$x^{p+p^2}/(1 - x^{1+p})$$

and the  $G(1)$ -form of 3.5 implies that  $\tilde{F}^0 \otimes X^{p^2-1}$  is a weak injective so  $\tilde{F}^0$  is 2-free.

To verify the claim, filter  $\tilde{F}^0$  by powers of  $(v_2, v_3)$ . Then  $E_0 \tilde{F}^0 \otimes X^{p^2-1}$  is an extended  $\overline{G(1)}$ -comodule with the appropriate  $\text{Ext}^0$ . This filtered  $\text{Ext}^0$  gives an upper bound on  $\text{Ext}_{G(1)}^0(A(1), \tilde{F}^0 \otimes X^{p^2-1})$ , while  $\text{Ext}_{G(1)}^0(A(1), \tilde{F}^0)$  is a lower bound, so the claim follows.

In  $\tilde{E}$  one has

$$r_{p^2} \left( \frac{v_3 v_2^j}{p v_1} \right) = \frac{v_2^{j+1}}{p v_1} = a_j r_{p^2} \left( \frac{v_2^{j+1+p}}{p v_1^{1+p}} \right)$$

where

$$a_j^{-1} = \binom{j + 1 + p}{p},$$

which is always nonzero in our range. Hence the element

$$u_j = \frac{v_3 v_2^j}{p v_1} - a_j \frac{v_2^{j+1+p}}{p v_1^{1+p}} \in \tilde{E}^0 \text{ is in } \ker r_{p^2}.$$

A similar argument works for  $\bar{E}$ , the group  $\text{Ext}_{G(1)}^0(A(1), \bar{E}^0 \otimes X^{p^2-1})$  is spanned by  $\{\gamma_i : i \geq 2\}$  where  $\gamma_i = v_3^i / p v_1 v_2$ . Hence we have proved

4.7. **THEOREM.** *The SES  $0 \rightarrow E \rightarrow \tilde{E} \rightarrow \bar{E} \rightarrow 0$  as above is a resolution of  $E$  by 2-free  $G(1)$ -comodules (4.4). Hence*

$$\text{Ext}(D_1^2 \otimes Y^{p^2-1}) = \hat{B} \oplus (\{\gamma_i : i \geq 2\} \oplus \{u_j : j \geq 0\} \otimes E(h_{20}) \otimes P(b_{20}))$$

where  $\hat{B}$  is the  $A(1)$ -module generated

$$\left\{ \bar{\beta}_{i+j/j} = \frac{v_2^{i+j}}{(i+j)p v_1^j} : i \geq 0, 1 \leq j \leq p \right\},$$

$$u_j = \frac{v_2^j v_3}{p v_1} - \frac{v_2^{j+p+1}}{\binom{j+p+1}{p} p v_1^{1+p}}$$

and

$$\gamma_i = \frac{v_3^i}{p v_1 v_2} \in \text{Ext}^1 \quad \square$$

Now we construct a SS to get from the Ext group above to  $\text{Ext}_{H(2)}(A(2), D_1^2 \otimes Y^{p-1})$ . We have an exact sequence

$$0 \longrightarrow Y^{p-1} \longrightarrow Y^{p^2-1} \xrightarrow{r_p} \Sigma^{p^2q} Y^{p-1} \longrightarrow 0$$

which defines  $b_{11} = \in \text{Ext}_{H(2)}^{2,p^2q}(Y^{p-1}, Y^{p-1})$ . By splicing this with itself repeatedly we get a LES

$$\begin{aligned} 0 \longrightarrow Y^{p-1} \longrightarrow Y^{p^2-1} \xrightarrow{r_p} \Sigma^{pq} Y^{p^2-1} \xrightarrow{r_{p^2-p}} \Sigma^{p^2q} Y^{p^2-1} \\ \xrightarrow{r_p} \Sigma^{(p+p^2)q} Y^{p^2-1} \dots \end{aligned}$$

tensoring this with  $D_1^2$  gives a 2-free resolution of  $D_1^2 \otimes Y^{p-1}$  and a SS as in 1.4, i.e.

**4.8. THEOREM.** *There is a SS converging to  $\text{Ext}(D_1^2 \otimes Y^{p-1})$  with  $E_1^{*,u} = E(h_{11}) \otimes P(b_{11}) \otimes \text{Ext}^u(D_1^2 \otimes Y^{p^2-1})$ ,  $h_{11} \in E_1^{1,0}$ ,  $b_{11} \in E_1^{2,0}$  and  $d_r: E_r^{s,u} \rightarrow E^{s+r,u-r+1}$ . Moreover  $d_1: E_1^{s,u} \rightarrow E_1^{s+1,u}$  is  $r_p$  for  $s$  even and  $r_{p^2-p}$  for  $s$  odd.*

*Proof.* We have done everything but compute  $d_1$ . Let  $x \in D_1^2 \otimes \Sigma^{2jp^2q} X^{p^2-1}$  have the form  $\Sigma x_i \otimes t_1^i$ . Then in the LES  $d(x) = \Sigma x_i \otimes r_p(t_1^i) = \Sigma \binom{i}{p} x_i \otimes t_1^{i-p}$ . If  $x$  is primitive then by the argument of 4.6  $x_i = (-1)^i r_i(x_0)$  so

$$\begin{aligned} d(x) &= \Sigma (-1)^i \binom{i}{p} r_i(x_0) \otimes t_1^{i-p} \\ &= \Sigma (-1)^i r_{i-p} r_p(x_0) \otimes t_1^{i-p} \end{aligned}$$

and this element corresponds to  $-r_p(x_0)$  under the isomorphism of 4.6. The argument for the case when the coboundary operator is  $r_{p^2-p}$  is similar. □

Now we compute  $d_1$  in 4.8, the  $E_1$ -term being determined by 4.7. We have (up to nonzero scalar multiplication)

$$\begin{aligned} r_p(u_i) &= (i+1) \frac{v_2^{i+p}}{pv_1^p}, & r_{p^2-p}(u_i) &= \binom{i+1}{p-1} \frac{v_2^{i+2}}{pv_1^2}, \\ r_p\left(\frac{v_2^j}{pv_1^j}\right) &= j \frac{v_2^{j-1}}{pv_1^{j-1}}, & r_{p^2-p}\left(\frac{v_2^j}{pv_1^j}\right) &= \binom{j}{p-1} \frac{v_2^{j+1-p}}{pv_1^{j+1-p}} \end{aligned}$$

$$r_p \left( \frac{v_2^{pj}}{p^2 v_1^i} \right) = j \frac{v_2^{pj-1}}{p v_1^{i-1}}, \quad \text{and}$$

$$r_{p^2-p} \left( \frac{v_2^{pj}}{p^2 v_1^i} \right) = j \frac{v_2^{pj+1-p}}{p v_1^{i+1-p}}.$$

Combining these observations with 4.7 and a routine calculation gives

4.9. **LEMMA.** *In the SS of 4.8  $d_1 : E_1^{s,u} \rightarrow E_1^{s+1,u}$  is trivial for  $u > 0$ , but nontrivial for  $u = 0$  such that  $E_2^{0,0}$  is generated by  $v_2^i/pjv_1, v_2^i/p^2v_1^i$  and  $u_{p,j-1}$  for  $j > 0, 1 \leq i \leq p; E_2^{1,0}$  by  $h_{11}(v_2^{pj}/p^2v_1^i)$  for  $i < p$  and  $h_{11}u_{j-1}$  for  $j \not\equiv 1 \pmod{p}; E_2^{2,0}$  by  $b_{11}(v_2^j/pjv_1)$  for  $j \not\equiv 1 \pmod{p}$  and  $b_{11}(v_2^{pj}/pv_1^i)$  for  $i \geq 3$ ; and for  $s > 2 E_2^{s,0} = b_{11}E_2^{s-2,0}$ .  $\square$*

Next observe that  $Y^{p-1}$  is a finitely generated  $A(2)$ -module with

$$\text{Hom}_{A(2)}(Y^{p-1}, A(2)) = \Sigma^{(1-p)q} Y^{p-1}$$

as  $H(2)$ -comodules. It follows that

$$\text{Ext}_{H(2)}(A(2), Y^{p-1} \otimes D_1^2) = \text{Ext}_{H(2)}(\Sigma^{(1-p)q} Y^{p-1}, D_1^2)$$

which is a module over the algebra  $\text{Ext}_{H(2)}(Y^{p-1}, Y^{p-1})$ . This contains the elements  $h_{11}$  and  $b_{11}$  represented by the exact sequences

$$0 \rightarrow Y^{p-1} \rightarrow Y^{2p-1} \rightarrow \Sigma^{pq} Y^{p-1} \rightarrow 0$$

and

$$0 \rightarrow Y^{p-1} \rightarrow Y^{p^2-1} \xrightarrow{r_p} \Sigma^{pq} Y^{p^2-1} \rightarrow \Sigma^{p^2q} Y^{p-1} \rightarrow 0$$

respectively. Hence we have

4.10. **PROPOSITION.** *If  $x \in E_2^{s,u}$  is a permanent cycle in the SS of 4.8 then so are the elements  $b_{11}^i x$  and  $h_{11} b_{11}^i x$  for all  $i \geq 0$ . Moreover multiplication by  $h_{11}$  and  $b_{11}$  commutes with differentials.  $\square$*

Now we can state our main computational result.

4.11. **THEOREM.** *In the SS of 4.8, the following differentials (along*

with those implied by 4.10) occur up to a nonzero scalar. All other differentials are trivial.

- (a)  $d_3(\gamma_i b_{20}^n) = n\gamma_i h_{11} b_{11} b_{20}^{n-1}$ .
- (b)  $d_3(\gamma_i h_{20} b_{20}^n) = n\gamma_i h_{20} h_{11} b_{11} b_{20}^{n-1}$ .
- (c)  $d_3(u_i b_{20}^n) = (n + i + 1)u_i h_{11} b_{11} b_{20}^{n-1}$  for  $n \geq 1$ .
- (d)  $d_{2p-2j-3}(u_{pi+j} h_{11} b_{20}^{p-2-j}) = b_{11}^{p-1-j} \beta_{pi+p/p-j}$  for  $0 \leq j \leq p - 3$ .
- (e)  $d_2(h_{20} u_i) = (i + 1)b_{11} \beta_{i+2}$  and  $d_3(h_{20} b_{20}^j u_i) = (i + j + 1)h_{11} b_{11} h_{20} b_{20}^{j-1} u_i$  for  $j \geq 1$ .
- (f)  $d_{2p-2j-2}(u_{pi+j} h_{11} h_{20} b_{20}^{p-2-j}) = \beta_{pi+p/(p-j-1,2)} h_{11} b_{11}^{p-j-1}$  for  $0 \leq j \leq p - 2$ .

Now we assume for simplicity that  $p = 5$ , the generalization to an arbitrary odd prime being straightforward.

We digress now to describe a simpler SS which serves as a paradigm for the one in question. Let  $P(1) = Z/(5)[t_1, t_2]/(t_1^{25}, t_2^5)$  with the evident Hopf algebra structure; it is isomorphic to the dual of the algebra generated by the Steenrod reduced powers  $P^1$  and  $P^5$ . Let  $\bar{Y}^n$  be the  $Z/(5)$ -vector space spanned by  $\{t_i^i : 0 \leq i \leq n\}$  for  $n < 25$ . We have  $\text{Ext}_{P(1)}(Z/(5), \bar{Y}^{24}) = \text{Ext}_{Q(1)}(Z/(5), Z/(5)) = E(h_{20}) \otimes P(b_{20})$  where  $Q(1) = Z/(5)[t_2]/[t_2^5]$ . The LES

$$0 \rightarrow \bar{Y}^4 \rightarrow \bar{Y}^{24} \xrightarrow{r_5} \Sigma^{5q} \bar{Y}^{24} \xrightarrow{r_{20}} \Sigma^{25q} \bar{Y}^{24} \rightarrow \dots$$

leads to a SS converging to  $\text{Ext}_{P(1)}(Z/(5), \bar{Y}^4)$  with  $E_2 = E(h_{11}, h_{20}) \otimes P(b_{11}, b_{20})$  with  $h_{11} \in E_2^{1,0}$ ,  $b_{11} \in E_2^{2,0}$ ,  $h_{20} \in E_2^{0,1}$ ,  $b_{20} \in E_2^{0,2}$  and  $d_r : E_r^{s,u} \rightarrow E_r^{s+r, u-r+1}$ . This SS is analogous to that of 4.8.

4.12. THEOREM. *The differentials in the above SS are as follows:*

- (a)  $d_3(b_{20}^i) = i h_{11} b_{11} b_{20}^{i-1}$ ;
- (b)  $d_9(h_{11} b_{20}^{5i+4}) = b_{11}^5 b_{20}^{5i}$ .

*These differentials commute with multiplication by  $h_{20}$ ,  $h_{11}$  and  $b_{11}$ , and all other differentials are trivial. Consequently  $\text{Ext}_{P(1)}(Z/(5), \bar{Y}^4)$  is a free module on  $P(b_{20}^5) \otimes E(h_{20})$  on  $\{b_{11}^i : 0 \leq i \leq 4\} \cup \{h_{11} b_{20}^i : 0 \leq i \leq 3\}$  with the product of  $h_{11}$  and  $h_{11} b_{20}^3$  being  $b_{11}^4$ .  $\square$*

Intuitively the elements  $h_{11} b_{20}^i$  are Massey products  $\langle h_{11}, \dots, h_{11}, b_{11}^i \rangle$  with  $i + 1$  factors of  $h_{11}$ . Hence the multiplicative extension  $h_{11} \cdot h_{11} b_{20}^3 = b_{11}^4$  follows from a Massey product identity,

$$h_{11} \langle h_{11}, h_{11}, h_{11}, h_{11}, b_{11}^3 \rangle = \langle h_{11}, h_{11}, h_{11}, h_{11}, h_{11} \rangle b_{11}^3 = b_{11}^4.$$

Since  $\bar{Y}^4$  is not a comodule algebra,  $\text{Ext}_{P(1)}(Z/(5), \bar{Y}^4)$  is not a ring, but we can get products by pulling back to  $\text{Ext}_{P(1)}(\bar{Y}^4, \bar{Y}^4)$ , which has a Yoneda product. We prefer to think of Massey products such as  $h_{11}b_{20}$  in the following way.  $b_{20} \in \text{Ext}_{P(1)}^2(Z/(5), \bar{Y}^{24})$  pulls back to an element in  $\text{Ext}_{P(1)}^2(Z/(5), \bar{Y}^5)$ , and  $h_{11} \in \text{Ext}_{P(1)}^1(Z/(5), \bar{Y}^4)$  pulls back to  $\text{Ext}_{P(1)}^1(\bar{Y}^5, Y^4)$ . The Yoneda product of these two elements is  $h_{11}b_{20} \in \text{Ext}_{P(1)}^3(Z/(5), \bar{Y}^4)$ . The choices made in the pullbacks are equivalent to the indeterminacy in the Massey product, which in this case is trivial.

Now let  $R^{s,t} = \text{Ext}_{P(1)}^{s,t}(Z/(5), \bar{Y}^4)$  as described above. 4.11 implies that  $\text{Ext}_{H(2)}(A(2), D_1^2)$ , after filtering to get rid of the elements  $\beta_{5i/(1,2)}$  of order 25, can be described as a direct sum of variously displaced copies of  $R$ , more precisely.

4.13. COROLLARY. *There is a (nonsplit) SES  $0 \rightarrow J \rightarrow \text{Ext}(D_1^2) \rightarrow K \rightarrow 0$  where  $J$  is the  $Z/(p)$ -vector space generated by  $\{\beta_i : i \equiv 0, 1 \pmod{p} : i > 1\}$   $K = K_1 \oplus K_2$  where  $K_1 = R \otimes \{\gamma_t : t \geq 2\}$  and  $K_2^{s,1} = \bigoplus_{i \geq 0} R^{s+2i+2, t+qi(p^2-1)}$ . Let  $i = pj + k$  with  $0 \leq k \leq p - 1$ . Then the  $i^{\text{th}}$  copy of  $R$  contains  $\beta_{i+2}$  and  $\beta_{pj+p/p-k}$  for  $0 \leq k \leq p - 3$ ,  $\beta_{pj+p/(1,2)}$  and  $\beta_{pj+p/2}$  for  $k = p - 2$  and  $u_{pj+p-1}$  for  $k = p - 1$ .  $\square$*

To visualize  $K_2$ , imagine a phantom generator in bidegree  $(-2i - 2, qi(p^2 - 1))$ , supporting a copy of  $R$  for each  $i \geq 0$ ;  $K_2$  consists of the portion of this object having nonnegative bidegree.

One can easily construct  $P(1)$ -comodules  $U_i$  satisfying

$$\text{Ext}_{P(1)}^s(Z/(5), U_i \otimes \bar{Y}^4) = \text{Ext}_{P(1)}^{s+2i}(Z/(5), \bar{Y}^4).$$

4.13 suggests that  $Y^4 \otimes D_1^2$  can be filtered in such a way that there is a SES

$$0 \rightarrow E_0(Y^4 \otimes D_1^2) \rightarrow \overline{H(2)} \square_{P(1)} V^0 \rightarrow \overline{H(2)} \square_{P(1)} V^1 \rightarrow 0$$

where  $\overline{H(2)} = H(2)/(5, v_1, v_2)$ ,  $V^0$  is a direct sum of copies of  $P(1)$  and  $\bar{Y}^4 \otimes U_i$  and  $V^1 = Y^4 \otimes \{\gamma_t : t \geq 2\}$ . Such a description of  $D_1^2$  would lead to a much more direct proof of 4.13, but we have been unable to construct the necessary isomorphism. Apparently it does not exist before one tensors with  $Y^4$ .

The proof of 4.12 is a model for the proof of 4.11 and we give the

former now for  $p = 5$ . We use the  $P(1)$  analogues of the following elementary facts which the reader can easily verify.

4.14. PROPOSITION.

(a) *There are pairings  $Y^i \otimes Y^j \rightarrow Y^{i+j}$  which induce pairings in Ext groups.*

(b) *The element  $h_{11} \in \text{Ext}(Y^4, Y^4)$  corresponding to  $0 \rightarrow Y^4 \rightarrow Y^9 \rightarrow \Sigma^{5q} Y^4 \rightarrow 0$  can be pulled back from a similar element in  $\text{Ext}^1(Y^{5i-5}, Y^4)$  for  $0 < i < 5$  given by  $0 \rightarrow Y^4 \rightarrow Y^{5i} \rightarrow \Sigma^{5q} Y^{5i-5} \rightarrow 0$ .*

(c)  *$b_{11} \in \text{Ext}^{2,25q}(Y^4, Y^4)$  corresponds to  $0 \rightarrow Y^4 \rightarrow Y^{24} \rightarrow \Sigma^{5q} Y^{24} \rightarrow \Sigma^{25q} Y^4 \rightarrow 0$ , which is obtained by splicing the SES's:*

$$0 \rightarrow Y^4 \rightarrow Y^{24} \rightarrow \Sigma^{5q} Y^{19} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Y^{19} \rightarrow Y^{24} \rightarrow \Sigma^{20q} Y^4 \rightarrow 0,$$

*so multiplication by  $b_{11}$  is the composite of the map  $\bar{h}_{11} : \text{Ext}^{s,t}(-, Y^4) \rightarrow \text{Ext}^{s+1,t+20q}(-, Y^{19})$  induced by the second SES and  $h_{11} : \text{Ext}^{s,t}(-, Y^{19}) \rightarrow \text{Ext}^{s+1,t+5q}(-, Y^4)$  induced by the first.*

(d) *The element  $b_{20} \in \text{Ext}_{G(2)}(A(2), A(2)) = \text{Ext}_{H(2)}(A(2), Y^\infty)$  pulls back to an element in  $\text{Ext}(Y^5)$  which maps to  $b_{11} \in \text{Ext}(A(2))$  under the map  $Y^5 \rightarrow \Sigma^{5q} A(2)$ . (A generalization to the elements  $u_i$  will be proved below in 4.15. The  $P(1)$  analogue is easy.)*

To prove 4.12(a), observe that 4.14(a) and (d) imply that  $b_{20}^i$  pulls back to  $\text{Ext}_{P(1)}^{2i}(Z/(5), \bar{Y}^{5i})$ . For  $i < 5$  the SES  $0 \rightarrow \bar{Y}^4 \rightarrow \bar{Y}^{5i} \rightarrow \Sigma^{5q} \bar{Y}^{5i-5} \rightarrow 0$  induces (by 4.14(b))

$$\begin{aligned} \text{Ext}_{P(1)}^{2i}(Z/(5), Y^{5i}) &\longrightarrow \text{Ext}_{P(1)}^{2i}(Z/(5), \Sigma^{5q} Y^{5i-5}) \\ &\xrightarrow{h_{11}} \text{Ext}_{P(1)}^{2i+1}(Z/5, \bar{Y}^4) \end{aligned}$$

under which  $b_{20}^i$  maps to  $b_{20}^{i-1} b_{11}$ , so  $h_{11} b_{11} b_{20}^{i-1} = 0$  and it must be killed by the indicated differential. Moreover for  $i < 4$  we can take the Yoneda product of  $b_{20}^i \in \text{Ext}_{P(1)}^{2i}(Z/(5), \bar{Y}^{5i})$  with the element in  $\text{Ext}_{P(1)}^1(\bar{Y}^{5i}, \bar{Y}^4)$  given by 4.14(b) and get an element in  $\text{Ext}_{P(1)}^{2i+1}(Z/5, \bar{Y}^4)$  representing  $h_{11} b_{20}^i$ , so the latter is a permanent cycle.

To see that multiplication by  $h_{20}$  commutes with differentials, note that  $h_{20}$  clearly pulls back to  $\text{Ext}_{P(1)}^{1,6q}(Z/(5), \bar{Y}^9)$  and the SES  $0 \rightarrow \bar{Y}^4 \rightarrow \bar{Y}^9 \rightarrow \Sigma^{5q} \bar{Y}^4 \rightarrow 0$  shows that the obstruction to pulling it back to  $\bar{Y}^4$  lies in  $\text{Ext}_{P(1)}^{1,q}(Z/(5), \bar{Y}^4) = 0$ . Hence we can use the pairing to pull  $b_{20}^i h_{20}$  back to  $Y^{5i+4}$  and argue as before.

Alternatively  $h_{20}$  can be realized as an element in  $\text{Ext}_{P(1)}^{1,6q}(\bar{Y}^4, \bar{Y}^4)$ , and there is a Yoneda pairing

$$\text{Ext}_{P(1)}(Z/(5), \bar{Y}^4) \otimes \text{Ext}_{P(1)}(\bar{Y}^4, \bar{Y}^4) \rightarrow \text{Ext}_{P(1)}(Z/(5), \bar{Y}^4)$$

For 4.12(b) we use the SES  $0 \rightarrow \bar{Y}^{19} \rightarrow \bar{Y}^{24} \rightarrow \Sigma^{20q} \bar{Y}^4 \rightarrow 0$  to get

$$\begin{aligned} \text{Ext}_{P(1)}^8(Z/(5), \bar{Y}^{24}) &\rightarrow \text{Ext}_{P(1)}^8(Z/(5), \Sigma^{20q} \bar{Y}^4) \\ &\xrightarrow{h_{11}} \text{Ext}_{P(1)}^9(Z/(5), \bar{Y}^{19}) \end{aligned}$$

in which  $b_{20}^4$  maps to  $b_{11}^4$  by 4.14(d), so  $\bar{h}_{11} b_{11}^4 = 0$  and  $b_{11}^5 = 0$  by 4.14(c). This forces  $d_9(h_{11} b_{20}^4) = b_{11}^5$  and  $b_{20}^5$  must be a permanent cycle. Multiplication by it gives the differentials of 4.12 for larger powers of  $b_{20}$ . This completes the proof of 4.12.

*Until further notice we will abbreviate  $\text{Ext}_{H(2)}(A(2), D_1^2 \otimes Y^n)$  by  $\text{Ext}(Y^n)$ .*

In order to prove 4.11 we need to generalize 4.14(d) to the elements  $u_i$  and  $u_i h_{20}$ . Let  $i = 5j + k$  with  $-1 \leq k \leq 3$ . Then an easy calculation shows  $(j + 1)r_{5k+5}(u_i) = v_2^{5j+5}/5v_1^{5-k}$  and  $r_{5m}(u_i) = 0$  for  $m > k + 1$ . From the SES

$$0 \rightarrow Y^{5m-1} \rightarrow Y^{24} \xrightarrow{r_{5m}} \Sigma^{5mq} Y^{24-5m} \rightarrow 0$$

we see that  $r_{5m}(u_i)$  is the obstruction to pulling  $u_i$  back to  $\text{Ext}(Y^{5m-1})$ .

Moreover if  $k \neq -1$  the obstruction  $\beta_{5j+5/5-k}$  pulls back to  $\text{Ext}^0(Y^0)$  which means that  $u_i$  pulls back to  $\text{Ext}(Y^{5k+5})$ . The element  $u_0 \in \text{Ext}^0(Y^{24})$  corresponds to  $b_{20} \in \text{Ext}_{H(2)}^2(A(2), Y^{24})$ , so this calculation verifies 4.14(d). Using the pairings of 4.14(a) we get a similar statement for  $b_{20}^n u_i$ . If  $k = -1$ ,  $u_i$  pulls back to  $\text{Ext}(Y^4)$  where we denote it by  $\beta_{5j+5/6}$ . Combining these we get

**4.15. LEMMA.** *With notation as above the element  $b_{20}^n u_i \in \text{Ext}(Y^{24})$  pulls back to an element in  $\text{Ext}(Y^{5k+5n+9})$  which projects to  $b_{11}^n \beta_{5j+5/5-k}$  under the map  $Y^{5k+5n+9} \rightarrow \Sigma^{5q(k+n+1)} Y^4$ .  $\square$*

The elements  $u_i h_{20}$  behave a little differently. A low dimensional calculation shows  $h_{20}$  pulls back to an element in  $\text{Ext}(Y^4)$  corresponding to  $v_2/5 - v_1^6/5^6$ ; there is some indeterminacy generated by  $v_1^6/5^5$ , but it will not affect our calculations. Applying  $r_4$  gives  $2v_1^2/5 = 2\alpha_2$ , which is a non-trivial obstruction to pulling back to  $Y^3$ .

Hence for  $0 \leq i \leq 3$   $u_i h_{20}$  pulls back to  $Y^{5i+4}$  and the obstruction to pulling back to  $Y^{5i+3}$  is a nonzero multiple of  $\alpha_2 \beta_{5/5-i} = \alpha_1 \beta_{5/4-i}$ . We want to compute the image under  $Y^{5i+4} \rightarrow \Sigma^{5iq} Y^4$ , which must be an element in  $\text{Ext}^{1, (36+i)q}(Y^4)$  whose image under  $r_4$  is  $\alpha_1 \beta_{5/4-i}$ . From 4.8(a) we see this element must be  $h_{11} \beta_{5/4-i, 2}$ .

Since  $u_4$  pulls back to  $Y^4$ ,  $h_{20} u_4$  pulls back to  $Y^9$  and the image under the map  $Y^9 \rightarrow \Sigma^{5q} Y^4$  lies in  $\text{Ext}^{1, 55q}(Y^4)$  which vanishes by 4.9, so  $u_4 h_{20}$  pulls back to  $Y^4$ .

The  $u_i h_{20}$  for  $i \geq 5$  behave similarly and we get

4.16. LEMMA. *Let  $i = 5j + k$  with  $-1 \leq k \leq 3$  as above. For  $k \neq -1$ ,  $h_{20} u_i b_{20}^n$  pulls back to  $\text{Ext}(Y^{5k+5n+9})$  and projects to  $h_{11} b_{11}^n \beta_{5j+5/4-k, 2}$  under the map  $Y^{5k+5n+9} \rightarrow \Sigma^{5q(k+n+1)} Y^4$ . For  $k = -1$ ,  $h_{20} u_{5j-1}$  pulls back to  $Y^4$  so  $b_{20}^n h_{20} u_{5j-1}$  pulls back to  $Y^{4+5n}$  where it projects to  $b_{11}^n h_{20} u_{5j-1}$ .  $\square$*

To get information about the SS from the previous two lemmas we have

4.17. LEMMA. *In the SS of 4.8*

(1) *if  $x \in E_2$  corresponds to an element in  $\text{Ext}(Y^{5i+4})$  for  $0 \leq i \leq 3$ , then  $h_{11} x$  is a permanent cycle;*

(b) *if it corresponds to one in  $\text{Ext}(Y^{5i+9})$  for  $0 \leq i \leq 3$  projecting under the map  $r_5: Y^{5i+9} \rightarrow \Sigma^{5q} Y^{5i+4}$  to an element corresponding to  $y \in E_2$ , then  $h_{11} y$  is the target of a differential; and*

(c) *if  $x \in E_2$  corresponds to an element in  $\text{Ext}(Y^{24})$  projecting under the map  $r_{20}: Y^{24} \rightarrow \Sigma^{20q} Y^4$  to an element corresponding to  $z$  then  $b_{11} z$  is the target of a differential.*

*Proof.* (a) The SES  $0 \rightarrow Y^4 \rightarrow Y^{5i+9} \rightarrow \Sigma^{5q} Y^{5i+4} \rightarrow 0$  has a connecting homomorphism sending  $x \in \text{Ext}(Y^{5i+4})$  to  $h_{11} x \in \text{Ext}(Y^4)$ .

(b) The above SES induces  $\text{Ext}(Y^{5i+9}) \rightarrow \text{Ext}(Y^{5i+4}) \xrightarrow{h_{11}} \text{Ext}(Y^4)$  in which  $x$  maps to  $y$  which is therefore annihilated by  $h_{11}$ .

(c) The SES  $0 \rightarrow Y^{19} \rightarrow Y^{24} \rightarrow \Sigma^{20q} Y^4 \rightarrow 0$  induces  $\text{Ext}(Y^{24}) \rightarrow \text{Ext}(Y^4) \xrightarrow{h_{11}} \text{Ext}(Y^{19})$  in which  $x$  maps to  $z$  which is therefore annihilated by  $\bar{h}_{11}$  and  $b_{11}$  (see 4.14(d)).  $\square$

We proved 4.12 by arguments similar to the above along with the observation that  $b_{20}^5$  is a permanent cycle because there is no possible target for a differential. In proving 4.11 we need to show the same is true of the elements  $b_{20}^{j-1} u_i$  and  $h_{20} b_{20}^{j-1} u_i$  with  $p \mid (i + j)$ ; this involves some book-keeping to be described in 4.19.

In applying 4.17(b) and (c) one would like to conclude that  $d_r(x) = h_{11}y$  and  $d_r(h_{11}x) = b_{11}z$  respectively; indeed this is the case in all of our examples. However this does not follow immediately because  $y$  and  $z$  are not necessarily uniquely determined by  $x$  since some choices are made in the construction. Equivalently,  $h_{11}y$  and  $b_{11}z$  could conceivably be killed by an earlier differential. Hence some care must be taken to verify the stated differentials.

We can however give a painless proof of 4.11(a) and (b), i.e. of the differentials involving the  $\gamma$ 's, by showing that each  $\gamma_k$  for  $k \geq 2$  induces a map of the SS of 4.12 into that of 4.8. Let  $K \subset D_1^2$  be the submodule spanned by  $\{(v_2^j v_3^k / 5v_1^{1+i}) : i, j, k \geq 0, k \geq 1 + i - j\}$ . It is easily seen to be a subcomodule.

Define a SES

$$(4.18) \quad 0 \rightarrow K \rightarrow \tilde{K} \rightarrow \bar{K} \rightarrow 0$$

by letting  $\tilde{K}$  have the same description as  $K$  without the condition  $j \geq 0$ . Then  $\bar{K}$  is spanned by  $\{(v_3^k / 5v_1^{1+i} v_2^{1+j}) : i, j \geq 0, k \geq 2 + i + j\}$ . An easy calculation gives

$$\text{Ext}_{H(2)}(A(2), \bar{K} \otimes Y^{24}) = \{\gamma_k : k \geq 2\} \otimes E(h_{20}) \otimes P(b_{20}).$$

Hence the SS for  $\text{Ext}_{H(2)}(A(2), \bar{K} \otimes Y^4)$  is simply  $\{\gamma_k : k \geq 2\}$  tensored with the one in 4.12. Now  $\gamma_k \in \text{Ext}_{H(2)}^0(A(2), \bar{K})$  which maps via (4.18) to  $\text{Ext}_{H(2)}^1(A(2), K)$  and thence to  $\text{Ext}_{H(2)}^1(A(2), D_1^2)$ , so 4.11(a) and (b) follow.

Note that this argument does *not* prevent a  $\gamma$ -related element from being hit by a differential originating elsewhere.

To prove 4.11(c)-(f) we describe the relation with 4.12 in more detail. The elements in 4.8 (excluding those involving  $\gamma$ 's) are analogous to those in 4.12 as indicated in the following table where  $0 \leq i \leq 3$

4.8 element	4.12 element
$v_2^{i+2} / (i + 2) 5v_1$	$h_{20} h_{11} b_{20}^i$
$\beta_{5/5-i}$	$b_{11}^{i+1}$
$\beta_{5/4-i, 2} h_{11}$	$h_{20} b_{11}^{1+i}$
$\beta_{5+i/5} h_{11}$	$h_{11} b_{20}^i b_{11}$
$u_j b_{20}^{k-1} h_{20}^\epsilon h_{11}^i$	$b_{20}^{j+k} h_{20}^\epsilon h_{11}^\tau$ for $\epsilon, \tau = 0, 1$

Hence the differentials in 4.11(c), (d), (e), and (f) correspond to those in 4.12 originating on  $b_{20}^{i+n+1}$ ,  $b_{20}^4 h_{11}$ ,  $h_{20} b_{20}^{i+j+1}$ , and  $h_{11} h_{20} b_{20}^4$  respectively. The vanishing of each target can be deduced from the previous 3 lemmas; as an instructive example we will show  $\beta_3 b_{11} = 0$  below. The survival of the elements corresponding to  $h_{11} b_{20}^i$  and  $h_{11} h_{20} b_{20}^i$  for  $0 \leq i \leq 3$  can be deduced from 4.17(a). This accounts for all except the elements corresponding to  $b_{20}^5$ . These are the only non  $\gamma$ -related elements said to be permanent cycles occurring in  $\text{Ext}^{s,t}$  with  $t/q$  divisible by 6, so it suffices to prove

4.19. LEMMA. *In the SS of 4.8 no element of the form  $u_i b_{20}^{j-1}$  with  $i + j$  divisible by  $p$  can support a nontrivial differential hitting a  $\gamma$ -related element.*

*Proof.* The possible targets are  $\gamma_k b_{11}^n$  and  $\gamma_k h_{20} h_{11} b_{20}^n$ . We will show that in our range none of these elements have the appropriate bigrading. To be safe we will not assume  $p = 5$ .  $u_i b_{20}^{j-1} \in \text{Ext}^{2j-2, q(p+1)(pj+i)} = \text{Ext}^{s,t}$  so  $t/q$  is divisible by  $p + 1$  and  $t/q - (s + 2)((p^2 - 1)/2) = p^2 j + p(i + j) + i - j(p^2 - 1) = (p + 1)(i + j) \equiv 0 \pmod{p + p^2}$ .

First  $\gamma_k b_{11}^n \in \text{Ext}^{1+2n, q(p^2(k+n)+(k-1)p+k-2)}$  so  $t/q \equiv p^2 n - p + k - 2 \equiv k + n - 1 \pmod{p + 1}$ . Since  $k \geq 2$ , this means  $k + n \geq p + 2$ , which puts the element out of our range.

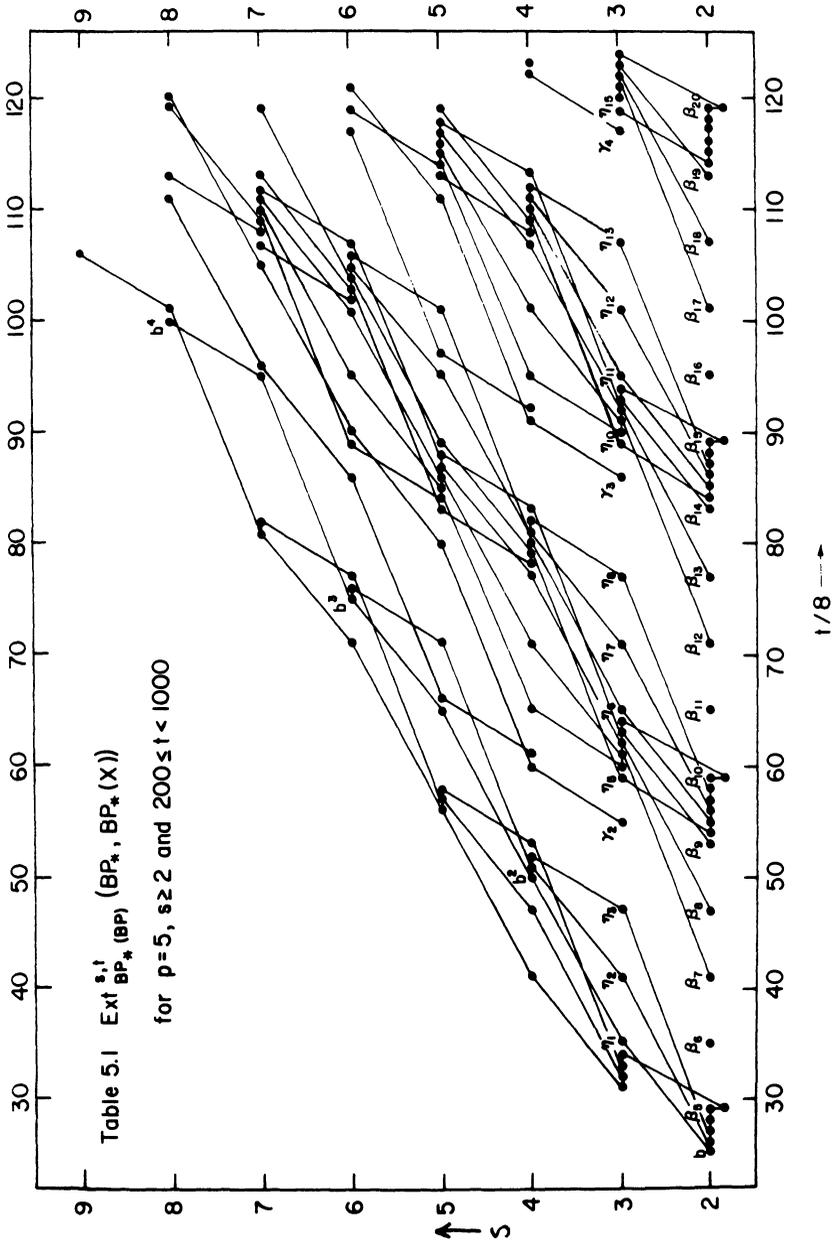
Next  $\gamma_k h_{20} h_{11} b_{20}^n \in \text{Ext}^{3+2n, q(p^2(k+n)+p(k+n-1)+m-1)}$ , giving  $t/q \equiv p + k - 1 \pmod{p + 1}$  so  $k = 2$ . This  $(s, t)$  must also satisfy  $t - (s - 3)(p^2 - 1)/2 \equiv 0 \pmod{p^2 + p}$ . But this quantity is  $(3 + n)(1 + p)$ , so  $n \geq p - 3$  which gives  $t/q = p^3 + 1$  which is again out of our range. □

Finally we describe, as a representative example, how to show  $b_{11} \beta_3 = 0$ . Consider the two maps of SES's,

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Sigma^{10q} Y^4 & \longrightarrow & \Sigma^{10q} Y^{19} & \longrightarrow & \Sigma^{15q} Y^{14} \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow r_{10} \\
 0 & \longrightarrow & \Sigma^{10q} Y^{14} & \longrightarrow & \Sigma^{10q} Y^{19} & \longrightarrow & \Sigma^{25q} Y^4 \longrightarrow 0
 \end{array}$$

and

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y^9 & \longrightarrow & Y^{14} & \xrightarrow{r_{10}} & \Sigma^{10q} Y^4 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & Y^9 & \longrightarrow & Y^{24} & \longrightarrow & \Sigma^{10q} Y^{k4} \longrightarrow 0.
 \end{array}$$



From this we get a commutative diagram

$$\begin{array}{ccccc}
 & & \text{Ext}^1(Y^{14}) & & \\
 & & \downarrow r_{10} & & \\
 \text{Ext}^0(\Sigma^{15q} Y^{14}) & \xrightarrow{h_{11}} & \text{Ext}^1(\Sigma^{10q} Y^4) & \longrightarrow & \text{Ext}^2(Y^9) \\
 \downarrow r_{10} & & \downarrow & & \parallel \\
 \text{Ext}^0(\Sigma^{25q} Y^4) & \longrightarrow & \text{Ext}^1(\Sigma^{10q} Y^{14}) & \longrightarrow & \text{Ext}^2(Y^9)
 \end{array}$$

in which the horizontal maps are the connecting homomorphisms and the top ‘ $L$ ’ is exact. Now  $r_{10}(h_{20}u_1) = h_{11}\beta_{5/3,2}$  by 4.16 and  $r_{10}\beta_{5/3,2} = \beta_3$  by 4.9. It follows that the image of  $\beta_3$  in  $\text{Ext}^2(Y^9)$  is trivial. But the bottom composite is multiplication by  $b_{11}$  composed with the inclusion  $Y^4 \rightarrow Y^9$ . It follows that  $b_{11}\beta_3$  must be in the kernel of  $h_{11}$ , and by inspection one sees that  $\text{Ext}^{1,37q}(Y^4) = 0$ , so this indeterminacy is trivial.

**5. Epilogue.** We have completed the calculation of  $\text{Ext}_{H(2)}(A(2), D_1^2 \otimes Y^{p-1})$  in our range of dimensions. The results for  $p = 5$  are shown on Table 5.1. Now it is a simple matter to compute  $\text{Ext}_{BP_*(BP)}(BP_*, T^{p-1})$ ; we will give the details below. Recall  $T^{p-1} = BP_*(X)$ , where  $X$  is a CW-spectrum with  $p$  cells,  $S^0 \cup e^q \cup e^{2q} \cup \dots \cup e^{(p-i)q}$  where each attaching map between adjacent cells is  $\alpha_1$ . Hence this Ext is the  $E_2$ -term of the ANSS for  $\pi_*(X)$ . Since there are no elements with filtration  $\geq 2p$ , the ANSS collapses by sparseness, so we have actually computed  $\pi_*(X)$ .

The real object of interest is of course the Ext for the sphere. To get at this we must study the SS analogous to that of 4.8 arising from the LES

$$0 \longrightarrow A(2) \longrightarrow Y^{p-1} \xrightarrow{r_1} \Sigma^q Y^{p-1} \xrightarrow{r_{p-1}} \Sigma^{pq} Y^{p-1} \longrightarrow \dots$$

We will describe this calculation elsewhere.

To compute  $\text{Ext}(T^{p-1})$  we use the complex  $D_1 \otimes Y^{p-1}$  as in section 3. Since  $D_1^i$  is a weak injective for  $i = 0, 1$ , the same is true of  $D_1^i \otimes Y^{p-1}$  and the skeletal filtration gives

$$E_0 \text{Ext}_{H(2)}^0(A(2), D_1^i \otimes Y^{p-1}) = \text{Ext}_{H(2)}^0(A(2), D_1^i) \otimes W^{p-1}$$

where  $W^{p-1}$  is the free  $Z_{(p)}$ -module on  $\{t_i^i : 0 \leq i \leq p - 1\}$ . For  $i = 0$  we get  $\text{Ext}^0$  spanned by  $\{v_1^k : 0 \leq k \leq p - 1\}$ . For  $i = 1$  the group extensions

are all nontrivial and  $\text{Ext}^{0,(pj+k)q}$  is generated by  $((v_1^p - p^p v_1^{-1} v_2)^j v_1^k / j p^{1+p})$  for  $0 \leq k \leq p - 1$ . It follows that in the SS of 3.1,  $d_1$

$$\left( \frac{v_1^p - v_1^{-1} v_2}{p^{1+p}} \right) = \frac{-v_2}{p v_1}$$

so  $\beta_1$  is dead and

$$\text{Ext}_{BP_*(BP_*)}^{s+2,t}(BP_*, T^{p-1}) = \text{Ext}_{H(2)s} t(A), D_1^2 \otimes Y^{p-1}) \quad \text{for } t > pq.$$

The following table displays this group for  $p = 5$  and  $25 \leq t/q < 125$ . Each dot represents a basis element. Vertical lines represent multiplication by 5 and horizontal lines represent the Massey product operation  $\langle -, 5, \alpha_1 \rangle$ , corresponding to multiplication by  $v_1$ . The diagonal lines correspond to operations  $\langle -, h_{11}, h_{11}, \dots, h_{11} \rangle$ .  $b$  denotes  $\beta_{5/5}$  and  $\eta_i$  denotes  $(v_2^{i-1}(v_2 t_2^5 + v_2^5 t_2 - v_2 t_1^{30} - v_3 t_1^5) / 5 v_1)$ . The generators to the left of the chart (i.e. with  $t < 200$ ) are  $\beta_i$  for  $i = 2, 3, 4$ , which would support diagonal lines hitting the generators of  $\text{Ext}^{3,8(29+i)}$ .

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REFERENCES

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[1] J. F. Adams, Stable homotopy and generalised homology, Part II, University of Chicago Press, 1974.

[2] M. Aubry, Calculs de groupes d'homotopie stables de la sphère par la suite spectral d'Adams-Novikov, to appear in *Math. Zeit.*

[3] H. R. Miller, D. C. Ravenel and W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, *Ann. Math.* **106** (1977), 469-516.

[4] O. Nakamura and S. Oka, Some differentials in the mod  $p$  Adams spectral sequence ( $p \geq 5$ ), *Hiroshima Math. J.* **6** (1976), 305-330; **7** (1977), 655-656.

[5] D. C. Ravenel, A novice's guide to the Adams-Novikov spectral sequence, *Lecture Notes in Math.* **658** (1978), 404-475, Springer-Verlag.