A solution to the Arf-Kervaire invariant problem

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Our main result
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Our main result

Our main theorem can be stated in three different but equivalent ways:

1. Manifold formulation: It says that a certain geometrically defined invariant \( \Phi(M) \) (the Arf-Kervaire invariant, to be defined later) on certain manifolds \( M \) is always zero.

2. Stable homotopy theoretic formulation: It says that certain long sought hypothetical maps between high dimensional spheres do not exist.

3. Unstable homotopy theoretic formulation: It says something about the EHP sequence (to be defined below), which has to do with unstable homotopy groups of spheres.

The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved.
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Our main result (continued)

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**Main Theorem**

The Arf-Kervaire elements $\theta_j \in \pi_{2j+1-2+n}(S^n)$ for large $n$ do not exist for $j \geq 7$. 
Our main result (continued)

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The $\theta_j$ in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial.
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The $\theta_j$ in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It has long been known that such things can exist only in dimensions that are 2 less than a power of 2.
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After 1980, the problem faded into the background because it was thought to be too hard. Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then.
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The Arf invariant of a quadratic form in characteristic 2

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$$q(x + y) = q(x) + q(y) + \lambda(x, y).$$
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In 1941 Arf proved that this invariant (along with the number $n$) determines the isomorphism type of $q$. 
On the money: Arf’s definition republished in 2009
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The Kervaire invariant of a framed \((4k + 2)\)-manifold

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The Kervaire invariant of a framed \((4k + 2)\)-manifold (continued)

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Kervaire defined a quadratic refinement \(q\) on its mod 2 reduction in terms of the trivialization of each sphere’s normal bundle. The Kervaire invariant \(\Phi(M)\) is defined to be the Arf invariant of \(q\).
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- Brown-Peterson (1966) showed that it vanishes for all positive even \(k\).
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- \(\theta_j\) is known to exist for \(1 \leq j \leq 5\), i.e., in dimensions 2, 6, 14, 30 and 62.
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- Our theorem says $\theta_j$ does *not* exist for $j \geq 7$. The case $j = 6$ is still open.
The EHP sequence

Assume all spaces in sight are localized and the prime 2. For each $n > 0$ there is a fiber sequence due to James,

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}.$$
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$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}.$$

This leads to a long exact sequence of homotopy groups

$$\cdots \to \pi_m(S^n) \xrightarrow{E} \pi_{m+1}(S^{n+1}) \xrightarrow{H} \pi_{m+1}(S^{2n+1}) \xrightarrow{P} \pi_{m-1}(S^n) \to \cdots$$
The EHP sequence (continued)

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Here \(E\) stands for Einhängung, the German word for suspension. \(H\) stands for Hopf invariant. \(P\) stands for Whitehead product.
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and we can ask about the image under $P$ of the generator of $\pi_{2n+1}(S^{2n+1})$. We denote it by $w_n \in \pi_{2n-1}(S^n)$, the **Whitehead square**.
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- When \( n \) is even, \( w_n \) it has infinite order and Hopf invariant two.
- \( w_n \) is trivial for \( n = 1, 3 \) and 7. In these cases \( w_{n+1} \in \pi_{2n+1}(S^{n+1}) \) is divisible by 2, the quotient having Hopf invariant one.
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- When $n$ is even, $w_n$ it has infinite order and Hopf invariant two.
- $w_n$ is trivial for $n = 1, 3$ and 7. In these cases $w_{n+1} \in \pi_{2n+1}(S^{n+1})$ is divisible by 2, the quotient having Hopf invariant one.
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- For such $n$, $w_n$ is divisible by 2 iff $n = 2^{j+1} - 1$ with $j > 2$ and $\theta_j$ exists, in which case $w_n = 2\theta_j$. 
The Hopf-Whitehead $J$ homomorphism

Let $SO(n)$ denote the special orthogonal group acting on $\mathbb{R}^n$. 

---

**Background and history**

Our main result

The Arf-Kervaire formulation

The unstable formulation

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**Our strategy**

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More precisely we have $H(w_{2s+1}^2 j - 1) = \beta_j$ for each $j > 0$ and $s \geq 0$. This result is essentially Adams' 1961 solution to the vector field problem.
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- *The Arf-Kervaire element* $\theta_j \in \pi_{2j+1-2}$ *exists for all* $j > 0$.  

---

**A solution to the Arf-Kervaire invariant problem**

Mike Hill
Mike Hopkins
Doug Ravenel

**Background and history**

- Our main result
- The Arf-Kervaire formulation
- The unstable formulation
- Questions raised by our theorem

**Our strategy**

- Ingredients of the proof
- The spectrum $\Omega$
- How we construct $\Omega$
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Ingredients of the proof (continued)

More ingredients of our proof:

- It uses complex cobordism theory. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory. It includes some highly developed computational techniques that began with work by Novikov and Quillen in the 60s. A pivotal tool in the subject is the theory of formal group laws.
- It also makes use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group $G$ (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers $\mathbb{Z}$, but by $\text{RO}(G)$, the real representation ring of $G$. Our calculations make use of this richer structure.
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(ii) **Periodicity Theorem.** It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of $k$ modulo 256.

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How we construct Ω (continued)

To get a $C_8$-spectrum, we use the following general construction for getting from a space or spectrum $X$ acted on by a group $H$ to one acted on by a larger group $G$ containing $H$ as a subgroup.
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In particular we get a $C_8$-spectrum

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In particular we get a $C_8$-spectrum

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This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.