Some early and middle mathematical work of Bob Stong

Doug Ravenel

University of Rochester

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Bob’s early career

► 1954-59: B.S. and M.A, University of Oklahoma
► 1959-62: M.S. and Ph.D. at University of Chicago under Dick Lashof
► 1962-64: Lieutenant in U.S. Army
► 1964-66: NSF Postdoctoral Fellow at Oxford
► 1966-68: Instructor at Princeton
► 1968 to present: University of Virginia
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Outline

Historical setting

Some early publications
- The connective covers paper
- The Stong-Hattori theorem
- Cobordism of maps
- Notes on Cobordism Theory

Some later papers
- Ochanine’s theorem
- Landweber-Stong 1988
- LRS 1993

The end

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The mathematical landscape in 1962

- Thom's work on cobordism, including the computation of $\pi_*(MO)$, the unoriented cobordism ring, 1954
- Lectures (at Princeton) on characteristic classes by Milnor, 1957 (later a book by Milnor and Stasheff, 1974)
- Bott periodicity, 1959
- The Adams spectral sequence, 1959, and Hopf invariant one, 1961
- The Riemann-Roch theorem of Atiyah-Hirzebruch, 1959
- Computations of $\pi_*(MU)$ (the complex cobordism ring) by Milnor and Novikov and of $\pi_*(MSO)$ (the oriented cobordism ring) by Wall, 1960
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- Brown-Peterson theory ($BP$), 1967
  - The Adams-Novikov spectral sequence, 1967
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  - Morava K-theory, roughly 1973
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Mod 2 cohomology of the connective covers of $BO$ and $BU$

“Determination of $H^*(BO(k,\cdots,\infty),\mathbb{Z}_2)$ and $H^*(BU(k,\cdots,\infty),\mathbb{Z}_2)$. “ appeared in the AMS Transactions in 1963.

It was a tour de force computation with the Serre spectral sequence and the Steenrod algebra.

It was the first determination of the cohomology of an infinite delooping of an infinite loop space, other than the Eilenberg-Mac Lane spectrum.

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The first three connective covers of $BO$ are

$$BO(2, \infty) = BSO$$
$$BO(3, \infty) = BO(4, \infty) = BSpin$$
$$BO(5, \infty) = \cdots = BO(8, \infty) = BString$$

The map from $H^\ast(BO)$ to the cohomology of each of these is onto.

This is not true of the higher connective covers.
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\[ BU(3, \infty) = BU(4, \infty) = BSU \]
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The integral cohomology of each of them is torsion free, which is not true of the higher connective covers.

The Thom spectra $MString$ and $MU(6)$ both figure in the theory of topological modular forms.
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The main result of the paper implies that

$$H^*(bo; \mathbb{Z}/2) = A \otimes_{A(1)} \mathbb{Z}/2$$

where $A(1)$ denotes the subalgebra of the mod 2 Steenrod algebra $A$ generated by $Sq^1$ and $Sq^2$.

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$$H^*(bu; \mathbb{Z}/2) = A \otimes_{Q(1)} \mathbb{Z}/2$$

where $Q(1)$ denotes the subalgebra generated by $Sq^1$ and $[Sq^1, Sq^2]$.

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Mod 2 cohomology of the connective covers of $BO$ and $BU$, 1963, methods used

It was known that

$$H^*(BO; \mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2, \ldots]$$

where $w_i \in H^i$ is the $i$th Stiefel-Whitney class.

Bob replaced the generator $w_i$ with $\theta_i \in H^i$, which he defined as the image of a certain Steenrod operation acting on $w_{2^m}$, where $m = \alpha(i - 1)$, with $\alpha(j)$ being the number of ones in the dyadic expansion of $j$. 
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For example we have $\theta_1 = w_1$,

\[ \theta_2 = w_2 \xrightarrow{Sq^1} \theta_3 \xrightarrow{Sq^2} \theta_5 \xrightarrow{Sq^4} \theta_9 \xrightarrow{Sq^8} \cdots \]

and

\[ \theta_4 = w_4 \xrightarrow{Sq^3} \theta_7 \xrightarrow{Sq^6} \theta_{13} \xrightarrow{Sq^{12}} \theta_{25} \xrightarrow{Sq^{24}} \cdots \]
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$$\xrightarrow{Sq^2} \theta_6 \xrightarrow{Sq^5} \theta_{11} \xrightarrow{Sq^{10}} \theta_{21} \xrightarrow{Sq^{20}} \cdots$$

$$\xrightarrow{Sq^4} \theta_{10} \xrightarrow{Sq^9} \theta_{19} \xrightarrow{Sq^{18}} \cdots$$

$$\xrightarrow{Sq^8} \theta_{18} \xrightarrow{Sq^{17}} \cdots$$

$$\xrightarrow{Sq^{16}} \cdots$$
Mod 2 cohomology of the connective covers of $BO$ and $BU$, 1963, methods used

He also makes use of some exact sequences of $A$-modules discovered by Toda. They can be obtained by applying the functor $A \otimes_{A(1)} (\cdot)$ to the following sequences exact of $A(1)$-modules.

For the $BU$ computation,

$$
\begin{align*}
A/A(Sq^3, Sq^1) & \xrightarrow{\cdot} \\
A/ASq^1 & \xrightarrow{\cdot} \\
\Sigma^3 A/A(Sq^3, Sq^1) & \xrightarrow{\cdot}
\end{align*}
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This short exact sequence defines (via a change-of-rings isomorphism relating $A(1)$ and $Q(1)$) an element in

\[\operatorname{Ext}^{1,3}_{Q(1)}(\mathbb{Z}/2, \mathbb{Z}/2)\]

corresponding to the complex form of Bott periodicity.
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The following 6-term exact sequence defines an element in

$$\text{Ext}^4_{A(1)}(\mathbb{Z}/2, \mathbb{Z}/2)$$

corresponding to the real form of Bott periodicity.
Mod 2 cohomology of the connective covers of $BO$ and $BU$, 1963, methods used

\[ A/A(Sq^2, Sq^1) \]
\[ A/ASq^1 \]
\[ \Sigma^2 A \]
\[ \Sigma^4 A \]
\[ \Sigma^7 A/ASq^1 \]
\[ \Sigma^{12} A/A(Sq^2, Sq^1) \]
Mod 2 cohomology of the connective covers of $BO$ and $BU$, 1963, methods used

Within this diagram are four modules of interest:

\[
\begin{align*}
A/A(Sq^1, Sq^2) & \quad A/ASq^2 \\
A/ASq^3 & \quad A/A(Sq^1, Sq^5)
\end{align*}
\]

which we denote by $M_0$, $M_1$, $M_2$ and $M_4$.

$M_s = H^*(bo\langle s \rangle)$, the mod 2 cohomology of the stable connective cover in the diagram

\[
bo = bo\langle 0 \rangle \quad bo\langle 1 \rangle \quad bo\langle 2 \rangle \quad bo\langle 4 \rangle
\]

\[
K(\mathbb{Z}, 0) \quad K(\mathbb{Z}/2, 1) \quad K(\mathbb{Z}/2, 2)
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Mod 2 cohomology of the connective covers of $BO$ and $BU$, 1963, methods used

Within this diagram are four modules of interest:

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Mod 2 cohomology of the connective covers of $BO$ and $BU$, statement of main result

**Theorem**

Let $s = 0, 1, 2$ or $4$, $8k + s > 0$,
and $K_{8k+s} = K(\pi_{8k+s}(BO), 8k + s)$.

Then

$$H^*(BO(k, \ldots, \infty)) = P(\theta_i : \alpha(i - 1) \geq 4k + s' - 1) \otimes H^*(K_{8k+s})/ (AQ_{8k+s})$$

where

$$(s', Q_s) = \begin{cases} (0, Sq^2) & \text{for } s = 0 \\ (1, Sq^2) & \text{for } s = 1 \\ (2, Sq^3) & \text{for } s = 2 \\ (3, Sq^5) & \text{for } s = 4 \end{cases}$$

The image of $H^*(K_{8k+s})$ is an unstable version of $M_s$. 
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The image of $H^*(K_{8k+s})$ is an unstable version of $M_s$. 

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- Historical setting
- Some early publications
  - The connective covers paper
  - The Stong-Hattori theorem
  - Cobordism of maps
  - Notes on Cobordism Theory
- Some later papers
  - Ochanine’s theorem
  - Landweber-Stong
  - LRS 1993
- The end
Mod 2 cohomology of the connective covers of $BO$ and $BU$, statement of main result

**Theorem**

Let $s = 0, 1, 2$ or $4$, $8k + s > 0$, and $K_{8k+s} = K(\pi_{8k+s}(BO), 8k + s)$.

Then

$$H^*(BO(k, \infty)) = P(\theta_i : \alpha(i - 1) \geq 4k + s' - 1) \otimes H^*(K_{8k+s})/(AQ_s \nu_{8k+s})$$

where

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The Stong-Hattori theorem

This was the subject of Bob’s two papers “Relations among characteristic numbers I and II,” which appeared in *Topology* in 1965 and 1966, while he was at Oxford.

He proved it for cobordism theories associated with five classical groups: $U$, $SU$, $SO$, $Spin$ and $Spin^c$.

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What the theorem says in the complex case

A complex manifold $M$ of dimension $n$ (meaning real dimension $2n$) comes equipped with a map $f : M \to BU$ classifying its stable normal bundle.

This gives us normal Chern classes $f^*(c_i) \in H^{2i}(M)$ for $1 \leq i \leq n$.

Any degree $n$ monomial $c^J$ in these can be evaluated on the fundamental homology class of $M$, and we get a Chern number

$$\langle c^J, [M] \rangle \in \mathbb{Z}.$$

Chern numbers were shown by Milnor to be complete cobordism invariants, i.e., $M$ is a boundary iff all of its Chern numbers vanish.
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Equivalently, the classifying map $f$ of the normal bundle on $M$ gives us an element $f_*([M]) \in H_{2n}(BU)$ which depends only on the cobordism class of $M$.

This leads to a monomorphically ring homomorphism

$$\text{complex cobordism ring} = \pi_*(MU) \to H_*(BU).$$

Composing this with the Thom isomorphism from $H_*(BU)$ to $H_*(MU)$ gives us the stable Hurewicz map

$$\eta : \pi_*(MU) \to H_*(MU)$$
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What is the image of $\eta : \pi_\ast(MU) \to H_\ast(MU)$?

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(i) Tensoring both side with the rationals converts $\eta$ into an isomorphism, so the image of $\eta$ has locally finite index.

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A reformulation of the Stong-Hattori theorem

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In the other four cases, $MSU$, $MSO$, $MSpin$ and $MSpin^c$, Bob proved similar statements using $KO_*$ instead of $K_*$. 
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Two maps $f_1 : M_1 \to N_1$ and $f_2 : M_2 \to N_2$ (where $M_1$ and $M_2$ are closed $m$-dimensional manifolds, while $N_1$ and $N_2$ are closed $n$-dimensional manifolds) are cobordant if there is a diagram

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\begin{array}{ccc}
M_1 & \xrightarrow{f_1} & V & \xleftarrow{g} & M_2 \\
\downarrow & & \downarrow & & \downarrow \\
N_1 & \xleftarrow{f_2} & W & \xrightarrow{g} & N_2
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where $V$ is an $(m + 1)$-manifold whose boundary is $M_1 \bigsqcup M_2$, and $W$ is an $(n + 1)$-manifold whose boundary is $N_1 \bigsqcup N_2$. 
Cobordism of maps, Topology 1966

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Assume that all manifolds in sight have compatible $G$-structures in their stable normal bundles for $G = O, SO, U$, etc.

Let $\Omega^G(m, n)$ denote the group of cobordism classes of maps as defined above, and let $MG$ denote the Thom spectrum associated with $G$.

Bob showed that

$$\Omega^G(m, n) \cong \lim_{r \to \infty} MG_n \left( \Omega^{r+m} MG_{r+n} \right)$$

$$\cong MG_n \left( \Omega^\infty MG_{n-m} \right)$$

The cobordism groups of maps are the bordism groups of the spaces in the $\Omega$-spectrum for $MG$.

He described these groups explicitly for $G = O$. 
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Cobordism of maps, Topology 1966, continued

One stop shopping for all your cobordism needs
Some early and middle mathematical work of Bob Stong

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Some early publications
- The connective covers paper
- The Stong-Hattori theorem
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Fast forward 20 years: Ochanine’s theorem on elliptic genera, 1987

**Theorem**

Suppose we have a homomorphism \( \varphi : \Omega_*^{SO} \to \Lambda \), from the oriented bordism ring to a commutative unital \( Q \)-algebra. Then it vanishes on all manifolds of the form \( \mathbb{C}P(\xi) \) with \( \xi \) an even-dimensional complex vector bundle over a closed oriented manifold if and only if the logarithm

\[
g(u) = \sum_{n \geq 0} \frac{\varphi(\mathbb{C}P^{2n})}{2n + 1} u^{2n+1}
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of the formal group law of \( \varphi \) is given by an elliptic integral of the first kind, i.e., by

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Such a homomorphism $\varphi : \Omega^\text{SO}_* \to \Lambda$ is now called an **elliptic genus**.

If $\delta = \varepsilon = 1$, then $\varphi$ is the signature.

If $\varepsilon = 0$ and $\delta = -\frac{1}{8}$, then $\varphi$ is the $\hat{A}$-genus.

We now know that $\Lambda$ need only be an algebra over the ring $\mathbb{Z}[1/2][\delta, \varepsilon]$, which can be interpreted as a ring of modular forms.

The manifolds on which $\varphi$ vanishes admit semi-free $S^1$-actions (since $\mathbb{C}P^{2n-1}$ does), and $\varphi(M)$ is an obstruction to the existence of such an action.

In the past 20 years there has been a lot of interest in interpreting such a genus geometrically or analytically.
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where $\hat{A}(M)$ the total $\hat{A}$-class of the tangent bundle of $M$, $\text{ch}(E)$ is the Chern character of the complexification of $E$ and $\rho_k(TM)$ is a certain element in $KO^*(M) \otimes \mathbb{Q}$, now known to be integral.
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In it we considered the genus defined above by Ochanine with values in $\mathbb{Z}[1/2][\delta, \varepsilon]$, regarded as a homomorphism out of the complex cobordism ring $MU_*$. Whenever one has an $R$-valued genus $\varphi$ on $MU_*$, one can ask if the functor

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The Ochanine genus $\varphi : MU_* \to \mathbb{Z}[1/2][\delta, \varepsilon]$ does not satisfy these criteria.

We showed that it becomes Landweber exact after inverting either $\varepsilon$ or $\delta^2 - \varepsilon$.

This means that if $R$ is the ring obtained from $\mathbb{Z}[1/2][\delta, \varepsilon]$ by inverting one or both of these elements, then the functor

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A homomorphism $\varphi : MU_* \rightarrow R$ is also equivalent (by Quillen’s theorem) to a 1-dimensional formal group law over $R$.

When $\varphi$ is the Ochanine genus, we get the formal group law associated with the elliptic curve defined by the Jacobi quartic,

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LRS and other elliptic curves

The same method entitles us to construct multiplicative homology theories with coefficient rings

\[ Z[\frac{1}{6}][g_2, g_3, \Delta^{-1}] \quad \text{where} \quad \Delta = g_2^3 - 27g_3^2 \]

corresponding to the elliptic curve defined by the Weierstrass equation

\[ y^2 = 4x^3 - g_2x - g_3, \]

and the ring

\[ Z[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}] \]

where \( \Delta \) is the discriminant of the equation

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\[ \mathbb{C}[\delta, \varepsilon] \text{ is naturally isomorphic to the ring } \mathcal{M}_*(\Gamma_0(2)) \text{ of modular forms for the group } \Gamma_0(2) \subset \text{SL}_2(\mathbb{Z}), \text{ with } \delta \text{ and } \varepsilon \text{ having weights 2 and 4, respectively.} \]

This isomorphism sends the subring \( \mathcal{M}_* = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon] \) to the modular forms whose \( q \)-expansions at the cusp \( \tau = \infty \) have coefficients in \( \mathbb{Z}[\frac{1}{2}] \).

Moreover, the localizations \( \mathcal{M}_*[\Delta^{-1}], \mathcal{M}_*[\varepsilon^{-1}] \) and \( \mathcal{M}_*[(\delta^2 - \varepsilon)^{-1}] \) correspond to the rings of modular functions which are holomorphic on \( \mathcal{H} \) (the complex upper half plane), \( \mathcal{H} \cup \{0\} \) and \( \mathcal{H} \cup \{\infty\} \), respectively, and whose \( q \)-expansions have coefficients in \( \mathbb{Z}[\frac{1}{2}] \).
The ring $\mathbb{C}[\delta, \varepsilon]$ is naturally isomorphic to the ring $M_\ast(\Gamma_0(2))$ of modular forms for the group $\Gamma_0(2) \subset \text{SL}_2(\mathbb{Z})$, with $\delta$ and $\varepsilon$ having weights 2 and 4, respectively. This isomorphism sends the subring $M_\ast = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ to the modular forms whose $q$-expansions at the cusp $\tau = \infty$ have coefficients in $\mathbb{Z}[\frac{1}{2}]$.

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\( \mathbb{C}[\delta, \varepsilon] \) is naturally isomorphic to the ring \( M_*(\Gamma_0(2)) \) of modular forms for the group \( \Gamma_0(2) \subset SL_2(\mathbb{Z}) \), with \( \delta \) and \( \varepsilon \) having weights 2 and 4, respectively.

This isomorphism sends the subring \( M_* = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon] \) to the modular forms whose \( q \)-expansions at the cusp \( \tau = \infty \) have coefficients in \( \mathbb{Z}[\frac{1}{2}] \).

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The end

Enjoy your retirement, Bob!