A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theory and Algebraic Geometry
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Mike Hill
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The spectrum $\Omega$
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Our main result

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- **Manifold formulation:** It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds $M$ is always zero.

- **Stable homotopy theoretic formulation:** It says that certain long sought hypothetical maps between high dimensional spheres do not exist.

- **Unstable homotopy theoretic formulation:** It says something about the EHP sequence (to be defined below), which has to do with unstable homotopy groups of spheres.

The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved.
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A wildly popular dance craze
Our main result (continued)

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**Main Theorem**

The Arf-Kervaire elements \( \theta_j \in \pi_{2j+1-2+n}(S^n) \) for large \( n \) do not exist for \( j \geq 7 \).
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**Main Theorem**

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The $\theta_j$ in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It has long been known that such things can exist only in dimensions that are 2 less than a power of 2.
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After 1980, the problem faded into the background because it was thought to be too hard. Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then.
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The Arf invariant of a quadratic form in characteristic 2

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In 1941 Arf proved that this invariant (along with the number $n$) determines the isomorphism type of $q$. 
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The Kervaire invariant of a framed $(4k + 2)$-manifold

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The \textit{Kervaire invariant} \(\Phi(M)\) is defined to be the Arf invariant of \(q\).
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Brown-Peterson (1966) showed that it vanishes for all positive even $k$. 
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- \(\theta_j\) is known to exist for \(1 \leq j \leq 5\), i.e., in dimensions 2, 6, 14, 30 and 62.
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- $\theta_j$ is known to exist for $1 \leq j \leq 5$, i.e., in dimensions 2, 6, 14, 30 and 62.

- Our theorem says $\theta_j$ does not exist for $j \geq 7$. The case $j = 6$ is still open.
Assume all spaces in sight are localized and the prime 2. For each \( n > 0 \) there is a fiber sequence due to James,

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S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}.
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This leads to a long exact sequence of homotopy groups

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Here \( E \) stands for Einhängung, the German word for suspension. \( H \) stands for Hopf invariant. \( P \) stands for Whitehead product.
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- $w_n$ is trivial for $n = 1, 3$ and 7. In these cases $w_{n+1} \in \pi_{2n+1}(S^{n+1})$ is divisible by 2, the quotient having Hopf invariant one.
- For other odd values of $n$, $H(w_{n+1}) = 2$ and $w_{n+1}$ is not divisible by 2, so $w_n$ has order 2.
The EHP sequence (continued)

For $m = 2n$ the sequence is

\[ \cdots \to \pi_{2n}(S^n) \xrightarrow{E} \pi_{2n+1}(S^{n+1}) \xrightarrow{H} \pi_{2n+1}(S^{2n+1}) \xrightarrow{P} \pi_{2n-1}(S^n) \to \cdots \]

and we can ask about the image under $P$ of the generator of $\pi_{2n+1}(S^{2n+1})$. We denote it by $w_n \in \pi_{2n-1}(S^n)$, the Whitehead square. The following facts are known about it.

- When $n$ is even, $w_n$ it has infinite order and Hopf invariant two.
- $w_n$ is trivial for $n = 1, 3$ and 7. In these cases $w_{n+1} \in \pi_{2n+1}(S^{n+1})$ is divisible by 2, the quotient having Hopf invariant one.
- For other odd values of $n$, $H(w_{n+1}) = 2$ and $w_{n+1}$ is not divisible by 2, so $w_n$ has order 2.
- For such $n$, $w_n$ is divisible by 2 iff $n = 2^{j+1} - 1$ with $j > 2$ and $\theta_j$ exists, in which case $w_n = 2\theta_j$. 
Let $SO(n)$ denote the special orthogonal group acting on $\mathbb{R}^n$. 

A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

Background and history

Our main result
The Arf-Kervaire formulation

The unstable formulation
Questions raised by our theorem

Our strategy
Ingredients of the proof
The spectrum $\Omega$
How we construct $\Omega$
Let $SO(n)$ denote the special orthogonal group acting on $\mathbb{R}^n$. Using the one point compactification, each element $g \in SO(n)$ induces a base point preserving map $S^n \to S^n$. 

The Hopf-Whitehead $J$ homomorphism
Let $SO(n)$ denote the special orthogonal group acting on $\mathbb{R}^n$. Using the one point compactification, each element $g \in SO(n)$ induces a base point preserving map $S^n \to S^n$. Thus we get a map $J : SO(n) \to \Omega^n S^n$ and for each $k > 0$ a homomorphism

$$\pi_k(SO(n)) \xrightarrow{J} \pi_k(\Omega^n S^n) = \pi_{n+k}(S^n).$$
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Both source and target known to be independent of $n$ for $n > k + 1$. 

The Hopf-Whitehead $J$ homomorphism
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$$\pi_k(SO) = \begin{cases} 
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Questions raised by our theorem

EHP sequence formulation. The World Without End

Hypothesis was the nicest possible statement of its kind given all that was known prior to our theorem. Now we know it cannot be true since $j$ does not exist for $j \geq 7$. This means the behavior of the indicated elements $P_j(j)$ for $j \geq 7$ is a mystery.

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Our method of proof offers a new tool for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future.
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\[ \pi_k(S^n) = \begin{cases} 0 & \text{if } k > n + 1 \\ \mathbb{Z} & \text{if } k = n + 1 \\ 0 & \text{if } k > n + 1 \end{cases} \]
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- It also makes use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group $G$ (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions.
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Background and history

Our main result
The Arf-Kervaire formulation
The unstable formulation
Questions raised by our theorem

Our strategy
Ingredients of the proof
The spectrum $\Omega$
How we construct $\Omega$
Ingredients of the proof (continued)

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The spectrum $\Omega$

We will produce a map $S^0 \to \Omega$, where $\Omega$ is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each $j$ is nontrivial. This means that if $j$ exists, we will see its image in $\ast(\Omega)$.

(ii) Periodicity Theorem. It is 256-periodic, meaning that $k(\Omega)$ depends only on the reduction of $k$ modulo 256.

(iii) Gap Theorem. $k(\Omega) = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.
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A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

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If $\theta_7 \in \pi_{254} (S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist.
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If $\theta_7 \in \pi_{254} (S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for $\theta_j$ for larger $j$ is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \pmod{256}$ for $j \geq 7$. 
Our spectrum $\Omega$ will be the fixed point spectrum for the action of $C_8$ (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$. 
How we construct $\Omega$

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How we construct $\Omega$ (continued)

To get a $C_8$-spectrum, we use the following general construction for getting from a space or spectrum $X$ acted on by a group $H$ to one acted on by a larger group $G$ containing $H$ as a subgroup.
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A general element of $G$ permutes these factors, each of which is left invariant by the subgroup $H$. In particular we get a $C_8$-spectrum $\tilde{\Omega} = \text{Map}_{C_2}(C_8, \text{MU})$. This spectrum is not periodic, but it has a close relative $\Omega$ which is.
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In particular we get a $C_8$-spectrum

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