Lecture 3: Equivariant stable homotopy theory

A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theory and Algebraic Geometry
Tokyo City University

November 5, 2009
Our strategy

Recall our goal is to prove

**Main Theorem**

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(i) It has an Adams-Novikov spectral sequence in which the image of each \( \theta_j \) is nontrivial.
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(iii) \( \pi_{-2}(\Omega) = 0 \).
Before we can describe any of this, we need to introduce equivariant stable homotopy theory.

**G-spaces**

A solution to the Arf-Kervaire invariant problem

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Equivariant stable homotopy theory

Peter May
John Greenlees
Gaunce Lewis

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Let $G$ be a finite group. A $G$-space is a topological space $X$ with a continuous left action by $G$; a based $G$-space is a $G$-space together with a basepoint fixed by $G$. We can convert an unbased $G$-spaces $X$ into a based one by taking the topological sum of $X$ and a $G$-fixed basepoint, denoted by $X^+$. 

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$MU$ as a $C_2$-spectrum

The norm functor

Our spectrum $\Omega$

$$\pi_*^u(MU^{(4)}_R)$$
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Equivariant stable homotopy theory

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G-CW complexes
Ordinary spectra
Equivariant spectra
$RO(G)$-graded homotopy

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Products and maps of $G$-spaces

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A map of $G$-spaces $f : X \to Y$ is said to be a weak $G$-equivalence if for each subgroup $H \subset G$, the induced map $f : X^H \to Y^H$ is a weak equivalence in the nonequivariant sense.
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For the orbit construction, given any subgroup \( H \) of \( G \) we may form the homogeneous space \( G/H \) and its based counterpart, \( G/H_+ \).

These are treated as 0-dimensional cells, and they play a role in equivariant theory analogous to the role of points in nonequivariant theory.
We form the $n$-dimensional cells from these homogeneous spaces. In the unbased context, the cell-sphere pair is

$$(G/H \times D^n, G/H \times S^{n-1})$$
**G-CW complexes via orbits (continued)**

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\[ \pi^U_*(MU_{\mathbb{R}}^{(4)}) \]
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Starting from these cell-sphere pairs, we form $G$-CW complexes exactly as nonequivariant CW-complexes are formed from the cell-sphere pairs $(D^n, S^{n-1})$. In such a complex, an element $\gamma \in G$ acts on a cell either by mapping it homeomorphically to another cell or by fixing it.
Let $V$ be an orthogonal representation of $G$. Denote its one-point compactification by $S^V$, with $\infty$ as the basepoint.
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We may also form the unit disc and unit sphere

$$D(V) = \{ v \in V : \|v\| \leq 1 \} \text{ and } S(V) = \{ v \in V : \|v\| = 1 \};$$

we think of them as unbased $G$-spaces.
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We can use these objects to build $G$-CW complexes as well. In this case $G$ can act on an individual cell by “rotating” it via the representation $V$. 
More general $G$-CW complexes

We can also mix these two constructions by considering cell-sphere pairs such as

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In such a complex, individual cells may be either permuted or rotated by an element of $G$. 
Toward equivariant spectra

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A prespectrum $D$ is a collection of spaces $\{D_n : n \gg 0\}$ with maps $\Sigma D_n \to D_{n+1}$. The adjoint of the structure map is a map $D_n \to \Omega D_{n+1}$.

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We get a spectrum $E = \{E_n : n \in \mathbb{Z}\}$ from the prespectrum $D$ by defining

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For technical reasons it is convenient to replace the collection $\{E_n\}$ by $\{E_V\}$ indexed by finite dimensional subspaces $V$ of a countably infinite dimensional real vector space $\mathcal{U}$ called a universe.
Toward equivariant spectra (continued)

The homotopy type of $E_V$ depends only on the dimension of $V$ and there are homeomorphisms

$$E_V \to \Omega^{|W|-|V|} E_W$$

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A map of spectra $f : E \to E'$ is a collection of maps of based $G$-spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps.
Let $G$ be a finite group. Experience has shown that in order to do equivariant stable homotopy theory, one needs $G$-spaces $E_V$ indexed by finite dimensional orthogonal representations $V$ sitting in a countably infinite dimensional orthogonal representation $\mathcal{U}$. 
$G$-equivariant spectra

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This universe $\mathcal{U}$ is said to be complete if it contains infinitely many copies of each irreducible representation of $G$. A canonical example of a complete universe for finite $G$ is the direct sum of countably many copies of the regular real representation of $G$. 
A $G$-equivariant spectrum (or $G$-spectrum for short) indexed on $\mathcal{U}$ consists of a based $G$-space $E_V$ for each finite dimensional subspace $V \subset \mathcal{U}$ together with a transitive system of based $G$-homeomorphisms

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for $V \subset W \subset \mathcal{U}$. Here $\Omega^V X = F(S^V, X)$ and $W - V$ is the orthogonal complement of $V$ in $W$. As in the classical case, the $G$-homotopy type of $E_V$ depends only on the isomorphism class of $V$. 

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Dropping the requirement that the structure maps be homeomorphisms gives us a $G$-prespectrum.
A solution to the Arf-Kervaire invariant problem

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\(G\)-spaces

\(G\)-CW complexes

Ordinary spectra

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3.13

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The structure map \(\tilde{\sigma}_{V,W}\) is adjoint to a map

\[\sigma_{V,W} : \Sigma^W V E_V \to E_W,\]

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where \( \Sigma^{V} X \) is defined to be \( S^{V} \wedge X \).

A suspension \( G \)-prespectrum is a \( G \)-prespectrum in which the maps above are \( G \)-equivalences for \( V \) sufficiently large.
Given a representation $V$ one has a suspension $G$-spectrum $\Sigma^\infty S^V$, which is often denoted abusively (as in the nonequivariant case) by $S^V$. 

$RO(G)$-graded homotopy groups
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As in the nonequivariant case, to define a prespectrum \( D \) it suffices to define \( G \)-spaces \( DV \) for a cofinal collection of representations \( V \).
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We define $S^{-V}$ by saying its $W$th space for $V \subset W$ is $S^{W-V}$. This is the analog of formal desuspension in the nonequivariant case.
**$RO(G)$-graded homotopy groups (continued)**

Given a virtual representation $\nu = W - V$, we define $S^\nu = \Sigma^W S^{-V}$. Hence we have a collection of sphere spectra graded over the orthogonal representation ring $RO(G)$.
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We define

\[
\pi^G_\nu(X) = [S^\nu, X]_G,
\]

the \( RO(G) \)-graded homotopy groups of the \( G \)-spectrum \( X \).
**MU as a $C_2$-spectrum**

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Our spectrum $\Omega$

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Since any orthogonal representation $V$ of $C_2$ is contained in $k\rho$ for $k \gg 0$, we can define the $C_2$-spectrum $MU$ by

$$MU_V = \lim_{\to k} \Omega^{k\rho - V} MU(k).$$
MU as a $C_2$-spectrum (continued)

This spectrum is known as real cobordism theory $MU_R$ and has been studied by Landweber, Araki, Hu-Kriz and Kitchloo-Wilson.

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Inducing and coinducing up to a larger group

Let $H \subset G$ be groups and let $X$ be a $H$-space. There are two ways to get a $G$-space from it. The corresponding functors are the left and right adjoints to the forgetful functor from $G$-spaces to $H$-spaces.
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An example is the the cell-sphere pair

$$(G/H \times D^n, G/H \times S^{n-1}).$$
Inducing and coinducing up to a larger group (continued)

There is also the coinduced $G$-space

$$\text{map}_H(G, X) = \{ f \in \text{map}(G, X) : f(\gamma \eta^{-1}) = \eta f(\gamma) \}$$

$\forall \eta \in H$ and $\gamma \in G$
Inducing and coinducing up to a larger group (continued)

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The underlying space here is the Cartesian product $X|^{G/H}$.
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There is a based analog of the coinduced $G$-space in which the underlying space is the smash product $X(|G/H|)$.

It extends to $H$-spectra. For a $H$-spectrum $X$ we denote the coinduced $G$-spectrum by $N^G_H X$, the norm of $X$ along the inclusion $H \subset G$. 
Norming up from $MU$

We apply this construction to the case $H = C_2$, $G = C_{2n+1}$ and $X = MU_R$. 

Let $\in G$ be a generator and let $z_i$ be a point in $MU_R$. Then the action of $G$ on $MU_R$ is given by

$$(z_1 \wedge \cdots \wedge z_{2n}) = z_{2n} \wedge z_1 \wedge \cdots \wedge z_{2n-1},$$

where $z_{2n}$ is the complex conjugate of $z_{2n}$. 

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We apply this construction to the case $H = C_2$, $G = C_{2^{n+1}}$ and $X = MU_R$. The underlying spectrum of $N^G_H MU_R$ is the $2^n$-fold smash power $MU^{(2^n)}$. 

Norming up from $MU$

We apply this construction to the case $H = C_2$, $G = C_{2n+1}$ and $X = MU_R$. The underlying spectrum of $N^G_H MU_R$ is the $2^n$-fold smash power $MU^{(2^n)}$.

Let $\gamma \in G$ be a generator and let $z_i$ be a point in $MU_R$. Then the action of $G$ on $MU_R^{(2^n)}$ is given by

$$\gamma(z_1 \wedge \cdots \wedge z_{2^n}) = \overline{z}_{2^n} \wedge z_1 \wedge \cdots \wedge z_{2^n-1},$$

where $\overline{z}_{2^n}$ is the complex conjugate of $z_{2^n}$. 

Our spectrum $\Omega$

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$\pi^U_* (MU^{(4)}_R)$
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\[ \pi_*^u(MU_R^{(4)}) \]
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In particular this makes $\mu_{4, R}$ into a $C_8$-spectrum. Our spectrum $\tilde{\Omega}$ is obtained from it by equivariantly inverting a certain element in its homotopy. Then $\Omega = \tilde{\Omega}^{C_8}$, which we will show to be equivalent to $\tilde{\Omega}^{hC_8}$.

The spectrum $\mu_{4, R}$ has two advantages over our earlier candidate $E_4$.

(i) It is a $C_8$-equivariant spectrum, while $E_4$ was merely an ordinary spectrum with a $C_8$ “action” for which a homotopy fixed point set could be defined.
**Our spectrum** $\tilde{\Omega}$

In particular this makes $MU^{(4)}_R$ into a $C_8$-spectrum. Our spectrum $\tilde{\Omega}$ is obtained from it by equivariantly inverting a certain element in its homotopy. Then $\Omega = \tilde{\Omega}^{C_8}$, which we will show to be equivalent to $\tilde{\Omega}^{hC_8}$.

The spectrum $MU^{(4)}_R$ has two advantages over our earlier candidate $E_4$.

(i) It is a $C_8$-equivariant spectrum, while $E_4$ was merely an ordinary spectrum with a $C_8$ "action" for which a homotopy fixed point set could be defined.

(ii) The action of $C_8$ on $\pi_* (MU^{(4)}_R)$ is transparent, unlike its mysterious action on $\pi_* (E_4)$. 

Our strategy (continued)

Our spectrum $\Omega$ will be derived from $MU_R^{(4)}$ regarded as a $C_8$-spectrum.
Our strategy (continued)

Our spectrum $\Omega$ will be derived from $MU^{(4)}_R$ regarded as a $C_8$-spectrum.

We need to describe the homotopy of the underlying nonequivariant spectrum, which we denote $\pi^u_*(MU^{(4)}_R)$. 
Recall that $H_*(MU; \mathbb{Z}) = \mathbb{Z}[b_i : i > 0]$ where $|b_i| = 2i$. $b_i$ is the image of a suitable generator of $H_{2i}(\mathbb{C}P^\infty)$ under the map

$$\Sigma^{\infty-2} \mathbb{C}P^\infty = \Sigma^{\infty-2} MU(1) \to MU.$$
Recall that $H_\ast(MU; \mathbb{Z}) = \mathbb{Z}[b_i : i > 0]$ where $|b_i| = 2i$. $b_i$ is the image of a suitable generator of $H_{2i}(\mathbb{C}P^\infty)$ under the map

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$$\pi_\ast^u(MU_R^{(4)})$$
Recall that $H_*(MU; \mathbb{Z}) = \mathbb{Z}[b_i : i > 0]$ where $|b_i| = 2i$. $b_i$ is the image of a suitable generator of $H_{2i}(CP^\infty)$ under the map

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It follows that $H_*(MU^{(4)}_R)$ is the 4-fold tensor power of this polynomial algebra. We denote its generators by $b_i(j)$ for $1 \leq j \leq 4$.

The action of $\gamma$ on these generators is given by

$$\gamma(b_i(j)) = \begin{cases} 
  b_i(j + 1) & \text{for } 1 \leq j \leq 3 \\
  (-1)^i b_i(1) & \text{for } j = 4.
\end{cases}$$
\[ \pi_*^u(MU^{(4)}_R) \] (continued)

\[ \pi_*^u(MU^{(4)}_R) \] is also a polynomial algebra with 4 generators in every positive even dimension.
\[ \pi_u^*(MU_R^{(4)}) \text{ (continued)} \]

\[ \pi_u^*(MU_R^{(4)}) \] is also a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension \(2i\) by \(r_i(j)\) for \(1 \leq j \leq 4\).
\[ \pi^u_*(MU_{R}^{(4)}) \] (continued)

\[ \pi^u_*(MU_{R}^{(4)}) \] is also a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension \(2i\) by \(r_i(j)\) for \(1 \leq j \leq 4\). The action of \(G = C_8\) is similar to that on the \(b_i(j)\), namely

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\( \pi^u_* (MU_R^{(4)}) \) is also a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension \( 2i \) by \( r_i(j) \) for \( 1 \leq j \leq 4 \). The action of \( G = C_8 \) is similar to that on the \( b_i(j) \), namely

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Earlier we said that \( \pi_* (MU_R) = \mathbb{Z}[x_i : i > 0] \) with \( |x_i| = 2i \).
$\pi_*^u(MU^{(4)}_R)$ is also a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension $2i$ by $r_i(j)$ for $1 \leq j \leq 4$. The action of $G = C_8$ is similar to that on the $b_i(j)$, namely

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Earlier we said that $\pi_*^u(MU_R) = \mathbb{Z}[x_i : i > 0]$ with $|x_i| = 2i$. We are using different notation now because $r_i(j)$ need not be the image of $x_i$ under any map $MU_R \to MU^{(4)}_R$. 
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\[ \pi^u_*(MU_R^{(4)}) \] (continued)
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\[ \pi^u_*\left(MU_R^{(4)}\right) \text{ (continued)} \]
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Here is some useful notation. For a subgroup $H \subset G$, let $h = |H|$, let $\rho_h$ denote its regular real representation and for $m \in \mathbb{Z}$, let

$$\tilde{S}(m\rho_h) = G_+ \wedge_H S^{m\rho_h}. \tag{3.25}$$

The underlying spectrum here is a wedge of $|G/H|$ copies of $S^{mh}$. We call this a slice cell of dimension $mh$.

We will explain how $\pi_*^u(MU^{(4)}_R)$ is related to maps from the $\tilde{S}(m\rho_h)$. The following notion is helpful.

**Definition**

Suppose $X$ is a $G$-spectrum such that its underlying homotopy group $\pi^u_k(X)$ is free abelian.
$$\pi_*^u(MU_R^{(4)})$$ (continued)

Here is some useful notation. For a subgroup $H \subset G$, let $h = |H|$, let $\rho_h$ denote its regular real representation and for $m \in \mathbb{Z}$, let

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**Definition**

Suppose $X$ is a $G$-spectrum such that its underlying homotopy group $\pi_k^u(X)$ is free abelian. A refinement of $\pi_k^u(X)$ is an equivariant map

$$c : \widehat{W} \to X$$

in which $\widehat{W}$ is a wedge of slice cells of dimension $k$ whose underlying spheres represent a basis of $\pi_k^u(X)$. 

Recall that in $\pi^u_*(MU_R)$, any monomial in the polynomial generators in dimension $2m$ is represented by an equivariant map from $S^{m\rho_2}$. 
Recall that in $\pi^u_*(MU_R)$, any monomial in the polynomial generators in dimension $2m$ is represented by an equivariant map from $S^{mp_2}$.

$\pi^u_*(MU(4))$ is a polynomial algebra with 4 generators in every positive even dimension.
Recall that in $\pi_*^u(MU_R^{(4)})$, any monomial in the polynomial generators in dimension $2m$ is represented by an equivariant map from $S^{m\rho_2}$.

$\pi_*^u(MU_R^{(4)})$ is a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension $2i$ by $r_i(j)$ for $1 \leq j \leq 4$. 

$\pi_*^u(MU_R^{(4)})$ (continued)
Recall that in $\pi_*^u(MU_R)$, any monomial in the polynomial generators in dimension $2m$ is represented by an equivariant map from $S^{mp_2}$.

$\pi_*^u(MU^{(4)}_R)$ is a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension $2i$ by $r_i(j)$ for $1 \leq j \leq 4$. The action of a generator $\gamma \in G = C_8$ is given by

$$\gamma(r_i(j)) = \begin{cases} r_i(j + 1) & \text{for } 1 \leq j \leq 3 \\ (-1)^i r_i(1) & \text{for } j = 4. \end{cases}$$
We will explain how $\pi_*^u(MU_R^{(4)})$ can be refined.
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$\pi^u_2(MU^{(4)})$ has 4 generators $r_1(j)$ that are permuted up to sign by $G$. 
We will explain how $\pi^u_\ast(MU^{(4)}_R)$ can be refined.

$\pi^u_2(MU^{(4)}_R)$ has 4 generators $r_1(j)$ that are permuted up to sign by $G$. It is refined by an equivariant map

$$\hat{W}_1 = \hat{S}(\rho_2) \rightarrow MU^{(4)}_R.$$
We will explain how $\pi_*^u(MU^{(4)})$ can be refined.

$\pi_2^u(MU^{(4)})$ has 4 generators $r_1(j)$ that are permuted up to sign by $G$. It is refined by an equivariant map

$$\hat{W}_1 = \hat{S}(\rho_2) \to MU^{(4)}.$$ 

Recall that the underlying spectrum of $\hat{W}_1$ is a wedge of 4 copies of $S^2$. 

$\pi_*^u(MU^{(4)}_{R})$ (continued)
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In π_4^u(MU^{(4)}) there are 14 monomials that fall into 4 orbits under the action of G, each corresponding to a map from a slice cell.

Recall that bS(4) is underlain by S_4 ∨ S_4. It follows that u_4^∗(MU^{(4)}) is refined by an equivariant map from cW_2 = bS(2^2) ∨ bS(2^2) ∨ bS(4) ∨ bS(2^2).
In $\pi_*^{u}(MU^{(4)}_R)$ there are 14 monomials that fall into 4 orbits under the action of $G$, each corresponding to a map from a slice cell.

$$\hat{S}(2 \rho_2) \leftrightarrow \{ r_1(1)^2, r_1(2)^2, r_1(3)^2, r_1(4)^2 \}$$
\[ \pi_*^{u}(MU^{(4)}_R) \] (continued)

In \( \pi_*^{u}(MU^{(4)}_R) \) there are 14 monomials that fall into 4 orbits under the action of \( G \), each corresponding to a map from a slice cell.

\[
\begin{align*}
\hat{S}(2\rho_2) & \leftrightarrow \{ r_1(1)^2, r_1(2)^2, r_1(3)^2, r_1(4)^2 \} \\
\hat{S}(2\rho_2) & \leftrightarrow \{ r_1(1)r_1(2), r_1(2)r_1(3), r_1(3)r_1(4), r_1(4)r_1(1) \}
\end{align*}
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\end{align*}
\]

(Recall that $\hat{S}(\rho_4)$ is underlain by $S^4 \vee S^4$.)
In $\pi^u_4(MU^{(4)})$ there are 14 monomials that fall into 4 orbits under the action of $G$, each corresponding to a map from a slice cell.

\[
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\end{align*}
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(Recall that $\hat{S}(\rho_4)$ is underlain by $S^4 \vee S^4$.) It follows that $\pi^u_4(MU^{(4)})$ is refined by an equivariant map from

\[
\hat{W}_2 = \hat{S}(2\rho_2) \vee \hat{S}(2\rho_2) \vee \hat{S}(\rho_4) \vee \hat{S}(2\rho_2).
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The refinement of $\pi_*^U(MU^{(4)})$ (continued)

A similar analysis can be made in any even dimension. $G$ always permutes monomials up to sign.
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Each group $\pi^u_{2n}(MU_R^{(4)})$ can be refined by a map from a wedge of slice cells $\hat{W}_n$. 
A similar analysis can be made in any even dimension. \( G \) always permutes monomials up to sign. The first case of a singleton orbit occurs in dimension 8, namely

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Each group \( \pi^u_{2n}(MU_\mathbb{R}^{(4)}) \) can be refined by a map from a wedge of slice cells \( \hat{W}_n \). Note that \( \hat{S}(m_{\rho_1}) \) never occurs as a wedge summand of \( \hat{W}_n \) because no monomial has a free orbit.