Lecture 6: The detection theorem

A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theory and Algebraic Geometry
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The Detection Theorem

$\theta_j$ in the Adams-Novikov spectral sequence

Formal $A$-modules

$\pi_\bullet(MU_k)$ and $R_\bullet$

The proof of the Detection Theorem

The proof of the Lemma
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It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \text{Ext}_{MU_* (MU)}^{2, 6i-2j} (MU_*, MU_*)$$

for certain values of $i$ and $j$. When $j = 1$, it is customary to omit it from the notation.
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for certain values of $i$ and $j$. When $j = 1$, it is customary to omit it from the notation. The definition of these elements can be found in Chapter 5 of the third author’s book *Complex Cobordism and Stable Homotopy Groups of Spheres*. 
Here are the first few of these in the relevant bidegrees.

\[ \theta_5 : \beta_{8/8} \text{ and } \beta_{6/2} \]
\[ \theta_6 : \beta_{16/16}, \beta_{12/4} \text{ and } \beta_{11} \]
\[ \theta_7 : \beta_{32/32}, \beta_{24/8} \text{ and } \beta_{22/2} \]
\[ \theta_8 : \beta_{64/64}, \beta_{48/16}, \beta_{44/4} \text{ and } \beta_{43} \]

and so on.
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and so on. In the bidegree of \( \theta_j \), only \( \beta_{2j-1/2j-1} \) has a nontrivial image (namely \( h_j^2 \)) in the Adams spectral sequence.
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$\theta_j$ in the Adams-Novikov spectral sequence (continued)

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We need to show that any element mapping to $h_j^2$ in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for $\Omega$. 

Detection Theorem

Let \( x \in \text{Ext}_{\text{MU}}^{2,2j+1} (\Omega \mu_{\bullet}, \mu) \) be any element whose image in \( \text{Ext}_{\text{A}}^{2,2j+1} (\mathbb{Z}/2, \mathbb{Z}/2) \) is \( h_j^2 \) with \( j \geq 6 \).
Detection Theorem

Let $x \in \text{Ext}^{2, 2^{j+1}}_{MU_* (MU)} (MU_*, MU_*)$ be any element whose image in $\text{Ext}^{2, 2^{j+1}}_A (\mathbb{Z}/2, \mathbb{Z}/2)$ is $h_j^2$ with $j \geq 6$. (Here $A$ denotes the mod 2 Steenrod algebra.)
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We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, the theory of formal \( A \)-modules, where \( A \) is the ring of integers in a suitable field.
Formal $A$-modules

Recall the a formal group law over a ring $R$ is a power series

$$F(x, y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x, y]]$$

with certain properties.
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For positive integers $m$ one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

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$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m][n](x) = [mn](x).$$
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These satisfy

$$[m + n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers $m$, and we get a homomorphism $\tau$ from $\mathbb{Z}$ to $\text{End}(F)$, the endomorphism ring of $F$. 
Formal $A$-modules (continued)

If the ground ring $R$ is an algebra over the $p$-local integers $\mathbb{Z}_p$ or the $p$-adic integers $\mathbb{Z}_p$, then we can make sense of $[m](x)$ for $m$ in $\mathbb{Z}_p$ or $\mathbb{Z}_p$. 
Formal $A$-modules (continued)

If the ground ring $R$ is an algebra over the $p$-local integers $\mathbb{Z}_{(p)}$ or the $p$-adic integers $\mathbb{Z}_p$, then we can make sense of $[m](x)$ for $m$ in $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$.

Now suppose $R$ is an algebra over a larger ring $A$, such as the ring of integers in a number field or a finite extension of the $p$-adic numbers.
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Now suppose $R$ is an algebra over a larger ring $A$, such as the ring of integers in a number field or a finite extension of the $p$-adic numbers. We say that the formal group law $F$ is a formal $A$-module if the homomorphism $\tau$ extends to $A$ in such a way that

$$[a](x) \equiv ax \bmod (x^2) \text{ for } a \in A.$$
Formal $A$-modules (continued)

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The theory of formal $A$-modules is well developed. Lubin-Tate used them to do local class field theory.
Formal $A$-modules (continued)

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$$\log_F(F(x, y)) = \log_F(x) + \log_F(y)$$

where

$$\log_F(x) = \sum_{n \geq 0} \frac{w^{2^n} - 1}{\pi^n} x^{2^n}.$$
Formal $A$-modules (continued)

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The classifying map $\lambda : MU_* \to R_*$ for $F$ factors through $BP_*$, where the logarithm is

$$\log(x) = \sum_{n \geq 0} \ell_n x^{2^n}.$$
Formal $A$-modules (continued)

Recall that $BP_* = \mathbb{Z}_2[v_1, v_2, \ldots]$ with $|v_n| = 2(2^n - 1)$. 
Recall that $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots]$ with $|v_n| = 2(2^n - 1)$. The $v_n$ and the $\ell_n$ are related by Hazewinkel’s formula,

\[
\ell_1 = \frac{v_1}{2}
\]
\[
\ell_2 = \frac{v_2}{2} + \frac{v_1^3}{4}
\]
\[
\ell_3 = \frac{v_3}{2} + \frac{v_1 v_2^2 + v_2 v_1^4}{4} + \frac{v_1^7}{8}
\]
\[
\ell_4 = \frac{v_4}{2} + \frac{v_1 v_3^2 + v_2 v_1^8}{4} + \frac{v_1^3 v_2^4 + v_1^9 v_2^2 + v_2 v_1^{12}}{8} + \frac{v_1^{15}}{16}
\]
\[
\vdots
\]
The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1}MU^{(4)}_R$?
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What does this have to do with our spectrum $\Omega = D^{-1}MU_R^{(4)}$?

Recall that $D = \Delta_1^{(8)} N_4^8 (\Delta_2^{(4)}) N_2^8 (\Delta_4^{(2)})$. 

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We will prove this later.
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Lemma

The classifying homomorphism $\lambda : \pi_\ast(MU) \to R_\ast$ for $F$ factors through $\pi_\ast(MU^{(4)}_R)$ in such a way that
The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1}MU^{(4)}_\mathbb{R}$? Recall that $D = \overline{\Delta}^{(8)}_1 N^8_4 (\overline{\Delta}^{(4)}_2) N^8_2 (\overline{\Delta}^{(2)}_4)$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

**Lemma**

*The classifying homomorphism* $\lambda : \pi_* (MU) \to R_*$ *for F factors through* $\pi_* (MU^{(4)}_\mathbb{R})$ *in such a way that*

- *the homomorphism* $\lambda^{(4)} : \pi_* (MU^{(4)}_\mathbb{R}) \to R_*$ *is equivariant,*
  where $C_8$ acts on $\pi_* (MU^{(4)}_\mathbb{R})$ as before, it acts trivially on $A$ and $\gamma w = \zeta_8 w$ for a generator $\gamma$ of $C_8$.  


The relation between $MU^{(4)}$ and formal $A$-modules

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- the homomorphism $\lambda^{(4)} : \pi_\ast(MU^4_R) \to R_\ast$ is equivariant, where $C_8$ acts on $\pi_\ast(MU^4_R)$ as before, it acts trivially on $A$ and $\gamma w = \zeta_8 w$ for a generator $\gamma$ of $C_8$.

- The element $D \in \pi_\ast(MU^4_R)$ that we invert to get $\Omega$ goes to a unit in $R_\ast$. 
The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1}MU^{(4)}_R$?

Recall that $D = \Delta^{(8)}_1 N^{8}_4(\overline{\Delta}^{(4)}_2) N^{8}_2(\overline{\Delta}^{(2)}_4)$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

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<thead>
<tr>
<th>Lemma</th>
</tr>
</thead>
<tbody>
<tr>
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We will prove this later.
The proof of the Detection Theorem

It follows that we have a map

\[ H^*(C_8; \pi_*(D^{-1}MU_R^{(4)})) = H^*(C_8; \pi_*(\Omega)) \to H^*(C_8; R_\ast). \]
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The source here is the $E_2$-term of the homotopy fixed point spectral sequence for $\Omega$, and the target is easy to calculate.
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Detection Theorem

Let $x \in \text{Ext}^{2,2}_{MU} \left( MU^*(MU), MU^*(MU) \right)$ be any element whose image in $\text{Ext}^{2,2}_{MU} (\mathbb{Z}/2, \mathbb{Z}/2)$ is $h_2^j$ with $j \geq 6$. (Here $A$ denotes the mod 2 Steenrod algebra.) Then the image of $x$ in $H^{2,2}_{C_8}(\Omega)$ is nonzero.

We will prove this by showing that the image of $x$ in $H^{2,2}_{C_8}(R_*^*)$ is nonzero.
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**Detection Theorem**

Let $x \in \text{Ext}_{MU_*^R(MU)}^{2,2j+1}(MU_*, MU_*)$ be any element whose image in $\text{Ext}^{2,2j+1}_A(\mathbb{Z}/2, \mathbb{Z}/2)$ is $h_j^2$ with $j \geq 6$. (Here $A$ denotes the mod 2 Steenrod algebra.) Then the image of $x$ in $H^{2,2j+1}(C_8; \pi_*(\Omega))$ is nonzero.
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It follows that we have a map

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We will prove this by showing that the image of $x$ in $H^{2,2i+1}(C_8; R_*)$ is nonzero.
The proof of the Detection Theorem (continued)

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$$BP_*(BP) = BP_*[t_1, t_2, \ldots] \text{ where } |t_n| = 2(2^n - 1).$$
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$$BP_* (BP) = BP_* [t_1, t_2, \ldots] \quad \text{where} \quad |t_n| = 2(2^n + 1).$$

We will abbreviate $\text{Ext}_{BP_* (BP)}^{s,t} (BP_*, BP_*)$ by $\text{Ext}^{s,t}$. 

Recall that

$$\text{Ext}_{BP_* (BP)}^{s,t} (BP_*, BP_*)$$

is a Hopf algebroid. There is a map from this Hopf algebroid to one associated with $H_* (\mathbb{C}^2; R)$ in which $t_n$ maps to an $R$-valued function on $\mathbb{C}^2$ (regarded as the group of 8th roots of unity) determined by

$$[x] = \sum_{n \geq 0} \langle t_n, \rangle x^{2n}.$$
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An easy calculation shows that the function $t_1$ sends a primitive root in $C_8$ to a unit in $R_*$. 
Let

\[ b_{1,j-1} = \frac{1}{2} \sum_{0 < i < 2^j} \binom{2^j}{i} \left[ t^j_i \mid t^{2^j-i}_1 \right] \in \text{Ext}^{2,2^j+1} \]
The proof of the Detection Theorem (continued)

Let

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It is known to be cohomologous to \( \beta_{2j-1}/2^{j-1} \) and to have order 2.
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It is known to be cohomologous to \( \beta_{2j-1}/2j-1 \) and to have order 2. We will show that its image in \( H^{2j,2j+1} (\mathbb{C}_8; R_*) \) is nontrivial for \( j \geq 2 \).
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$$b_{1,j-1} = \frac{1}{2} \sum_{0 < i < 2^j} \binom{2^j}{i} \left[ t^i_1 | t^{2^j-i}_{1}\right] \in \text{Ext}^{2,2^{j+1}}$$

It is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2. We will show that its image in $H^{2,2^{j+1}}(C_8; R_*)$ is nontrivial for $j \geq 2$.

$H^*(C_8; R_*)$ is the cohomology of the cochain complex

$$R_*[C_8] \xrightarrow{\gamma^{-1}} R_*[C_8] \xrightarrow{\text{Trace}} R_*[C_8] \xrightarrow{\gamma^{-1}} \cdots$$

where Trace is multiplication by $1 + \gamma + \cdots + \gamma^7$. 

The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in $s$ with period 2.
The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in $s$ with period 2. We have

$$H^1(C_8; R_{2m}) = \ker (1 + \zeta_8^m + \cdots + \zeta_8^{7m})/\im (\zeta_8^m - 1)$$
The proof of the Detection Theorem (continued)

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$$H^1(C_8; R_{2m}) = \ker (1 + \zeta_8^m + \cdots + \zeta_8^{7m})/\im (\zeta_8^m - 1)$$

$$= \begin{cases} 
  w^m A/(\pi) & \text{for } m \text{ odd} \\
  w^m A/(\pi^2) & \text{for } m \equiv 2 \mod 4 \\
  w^m A/(2) & \text{for } m \equiv 4 \mod 8 \\
  0 & \text{for } m \equiv 0 \mod 8 
\end{cases}$$
The proof of the Detection Theorem (continued)

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\end{cases}$$

$$H^2(C_8; R_{2m}) = \ker (\zeta_8^m - 1)/\text{im} (1 + \zeta_8^m + \cdots + \zeta_8^{7m})$$
The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_\ast)$ for $s > 0$ are periodic in $s$ with period 2. We have

$$H^1(C_8; R_{2m}) = \ker (1 + \zeta_8^m + \cdots + \zeta_8^{-7m})/\text{im} (\zeta_8^m - 1)$$

$$= \begin{cases} 
  w^m A/(\pi) & \text{for } m \text{ odd} \\
  w^m A/(\pi^2) & \text{for } m \equiv 2 \text{ mod } 4 \\
  w^m A/(2) & \text{for } m \equiv 4 \text{ mod } 8 \\
  0 & \text{for } m \equiv 0 \text{ mod } 8 
\end{cases}$$

$$H^2(C_8; R_{2m}) = \ker (\zeta_8^m - 1)/\text{im} (1 + \zeta_8^m + \cdots + \zeta_8^{-7m})$$

$$= \begin{cases} 
  w^m A/(8) & \text{for } m \equiv 0 \text{ mod } 8 \\
  0 & \text{otherwise}
\end{cases}$$
The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_\ast)$ for $s > 0$ are periodic in $s$ with period 2. We have

$$H^1(C_8; R_{2m}) = \ker (1 + \zeta_8^m + \cdots + \zeta_8^{-7m})/\text{im} (\zeta_8^m - 1) = \begin{cases} w^m A/(\pi) & \text{for } m \text{ odd} \\ w^m A/(\pi^2) & \text{for } m \equiv 2 \mod 4 \\ w^m A/(2) & \text{for } m \equiv 4 \mod 8 \\ 0 & \text{for } m \equiv 0 \mod 8 \end{cases}$$

$$H^2(C_8; R_{2m}) = \ker (\zeta_8^m - 1)/\text{im} (1 + \zeta_8^m + \cdots + \zeta_8^{-7m}) = \begin{cases} w^m A/(8) & \text{for } m \equiv 0 \mod 8 \\ 0 & \text{otherwise} \end{cases}$$

An easy calculation shows that $b_{1,j-1}$ maps to $4w^{2j}$, which is the element of order 2 in $H^2(C_8; R_{2j+1})$. 
To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. 
The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\left\{ \beta c(j, k) / 2^{j-1-2k} : 0 \leq k < j/2 \right\}$$

where $c(j, k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. 
The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\{ \beta_{c(j,k)/2^{j-1-2k}} : 0 \leq k < j/2 \}$$

where $c(j,k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k > 0$ map to zero.
The proof of the Detection Theorem (continued)

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We will see in the proof of the Lemma below that $\nu_1$ and $\nu_2$ map to unit multiples of $\pi^3w$ and $\pi^2w^3$ respectively.
The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\{ \beta_{c(j,k)2^{j-1-2k}} : 0 \leq k < j/2 \}$$

where $c(j,k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. Note that $\beta_{c(j,0)2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k > 0$ map to zero.

We will see in the proof of the Lemma below that $\nu_1$ and $\nu_2$ map to unit multiples of $\pi^3 w$ and $\pi^2 w^3$ respectively. This means we can define a valuation on $BP_*$ compatible with the one on $A$ in which $\|2\| = 1$, $\|\pi\| = 1/4$, $\|\nu_1\| = 3/4$ and $\|\nu_2\| = 1/2$. 
To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\{ \beta_{c(j,k)}/2^{j-1-2k} : 0 \leq k < j/2 \}$$

where $c(j, k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k > 0$ map to zero.

We will see in the proof of the Lemma below that $\nu_1$ and $\nu_2$ map to unit multiples of $\pi^3 w$ and $\pi^2 w^3$ respectively. This means we can define a valuation on $BP_*$ compatible with the one on $A$ in which $\|2\| = 1$, $\|\pi\| = 1/4$, $\|\nu_1\| = 3/4$ and $\|\nu_2\| = 1/2$. We extend the valuation on $A$ to $R_*$ by setting $\|w\| = 0$. 

The proof of the Detection Theorem (continued)
Hence for $k \geq 1$ and $j \geq 6$ we have

$$\left\| \beta_c(j,k) / 2^{j-1-2k} \right\|$$
The proof of the Detection Theorem (continued)

Hence for \( k \geq 1 \) and \( j \geq 6 \) we have

\[
\| \beta_{c(j,k)}/2^{i-1-2k} \| = \left\| \frac{v_2^{c(j,k)}}{2v_1^{2i-1-2k}} \right\|
\]
The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

\[
\left\| \beta \frac{c(j, k)}{2^{i-1} - 2k} \right\| = \left\| \frac{v_2^{c(j, k)}}{2v_1^{2^{i-1} - 2k}} \right\|
\]

\[
= \frac{c(j, k)}{2} - \frac{3 \cdot 2^{j-1 - 2k}}{4} - 1
\]
The proof of the Detection Theorem (continued)

Hence for \( k \geq 1 \) and \( j \geq 6 \) we have

\[
\left\| \beta_{c(j,k)}/2^{j-1-2k} \right\| = \left\| \frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}} \right\| = \frac{c(j, k)}{2} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 = \frac{2^j + 2^{j-1-2k}}{6} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 = (2^{j-1} - 7 \cdot 2^{j-3-2k})/3 - 1 \geq 5.
\]
The proof of the Detection Theorem (continued)

Hence for \( k \geq 1 \) and \( j \geq 6 \) we have

\[
\left| \beta_{c(j,k)}/2^{j-1-2k} \right| = \left| \frac{\nu_2^{c(j,k)}}{2 \nu_1^{2^{j-2}} - 1} \right|
\]

\[
= \frac{c(j, k)}{2} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1
\]

\[
= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1
\]

\[
= \frac{(2^{j-1} - 7 \cdot 2^{j-3-2k})/3 - 1}{3 - 1}
\]

\[
\geq 5.
\]

This means \( \beta_{c(j,k)}/2^{j-1-2k} \) maps to an element that is divisible by 8 and therefore zero.
The proof of the Detection Theorem (continued)

We have to make a similar computation with the element
\( \alpha_1 \alpha_{2j-1} \).
The proof of the Detection Theorem (continued)

We have to make a similar computation with the element \( \alpha_1 \alpha_{2j-1} \). We have

\[
\| \alpha_{2j-1} \| = \left\| \frac{v_{2j-1}}{2} \right\| = \frac{3(2^j - 1)}{4} - 1
\]

\[
\geq \frac{21}{4} - 1 \geq 4 \quad \text{for } j \geq 3.
\]
The proof of the Detection Theorem (continued)

We have to make a similar computation with the element $\alpha_1 \alpha_2^{j-1}$. We have

$$\left\| \alpha_2^{j-1} \right\| = \left\| \frac{v_1^{2j-1}}{2} \right\| = \frac{3(2^j - 1)}{4} - 1$$

$$\geq \frac{21}{4} - 1 \geq 4 \quad \text{for } j \geq 3.$$ 

This completes the proof of the Detection Theorem modulo the Lemma.
The proof of the Lemma

Here it is again.
The proof of the Lemma

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**Lemma**

The classifying homomorphism \( \lambda : \pi_*(MU) \rightarrow R_* \) for \( F \) factors through \( \pi_*(MU^{(4)}_R) \) in such a way that
The proof of the Lemma

Here it is again.

**Lemma**

The classifying homomorphism \( \lambda : \pi_* (MU) \rightarrow R_* \) for \( F \) factors through \( \pi_* (MU_R^{(4)}) \) in such a way that

- the homomorphism \( \lambda^{(4)} : \pi_* (MU_R^{(4)}) \rightarrow R_* \) is equivariant, where \( C_8 \) acts on \( \pi_* (MU_R^{(4)}) \) as before, it acts trivially on \( A \) and \( \gamma w = \zeta_8 w \) for a generator \( \gamma \) of \( C_8 \).
The proof of the Lemma

Here it is again.

**Lemma**

*The classifying homomorphism* \( \lambda : \pi_* (MU) \to R_* \) *for* \( F \) *factors through* \( \pi_* (MU_{R}^{(4)}) \) *in such a way that*

- *the homomorphism* \( \lambda^{(4)} : \pi_* (MU_{R}^{(4)}) \to R_* \) *is equivariant, where* \( C_8 \) *acts on* \( \pi_* (MU_{R}^{(4)}) \) *as before, it acts trivially on* \( A \) *and* \( \gamma w = \zeta_8 w \) *for a generator* \( \gamma \) *of* \( C_8 \).

- *The element* \( D \in \pi_* (MU_{R}^{(4)}) \) *that we invert to get* \( \Omega \) *goes to a unit in* \( R_* \).
The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring $K$.
The proof of the Lemma (continued)

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```
```

$\eta_L \quad \parallel \quad \eta_R$

$\pi_*(MU) \quad \quad \quad \pi_*(MU) \quad \quad \quad \pi_*(MU)$

$\lambda_1 \quad \lambda (2) \quad \lambda_2$

$\lambda (2)$

$\pi_*(MU(2))$

$MU_*(MU)$

$\pi_*(MU_R)$

$K$

The maps $\eta_L$ and $\eta_R$ classify two formal group laws $F_1$ and $F_2$ over $K$. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda (2)$ is equivalent to that of a compatible strict isomorphism between $F_1$ and $F_2$. 
The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring $K$.

\[
\begin{array}{ccc}
\pi_*(MU) & & \pi_*(MU_R) \\
| & & \\
\eta_L & & \eta_R \\
| & & \\
MU_*(MU) & & \pi_*(MU) \\
& & \\
\lambda_1 & \lambda^{(2)} & \lambda_2 \\
\downarrow & \Downarrow & \downarrow \\
K & & K
\end{array}
\]

The maps $\lambda_1$ and $\lambda_2$ classify two formal group laws $F_1$ and $F_2$ over $K$. 

The maps $\lambda_1$ and $\lambda_2$ classify two formal group laws $F_1$ and $F_2$ over $K$. 

A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

The Detection Theorem

$\theta_j$ in the Adams-Novikov spectral sequence
Formal $A$-modules
$\pi_*(MU_R)$ and $R_*$
The proof of the Detection Theorem

The proof of the Lemma
The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring $K$.

\[
\begin{array}{ccc}
\pi_* (MU) & \rightarrow & \pi_* (MU) \\
\downarrow^{\lambda_1} & & \downarrow^{\lambda_2} \\
K & \rightarrow & \pi_* (MU_R) \\
& & \downarrow^{\lambda(2)} \\
& & \pi_* (MU_R^{(2)}) \\
& & \downarrow^{\eta_L} \\
& & \eta_R \\
& & \pi_* (MU) \\
\end{array}
\]

The maps $\lambda_1$ and $\lambda_2$ classify two formal group laws $F_1$ and $F_2$ over $K$. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws.
The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring $K$.

$$
\begin{array}{ccc}
\pi_*(MU) & \rightarrow & \pi_*(MU) \\
\downarrow & & \downarrow \\
\pi_*(MU^{(2)}) & \rightarrow & \pi_*(MU) \\
\downarrow & & \downarrow \\
K & \rightarrow & K
\end{array}
$$

The maps $\lambda_1$ and $\lambda_2$ classify two formal group laws $F_1$ and $F_2$ over $K$. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a compatible strict isomorphism between $F_1$ and $F_2$. 
The proof of the Lemma (continued)

Similarly consider the diagram

\[
\begin{array}{ccc}
\pi_* (MU) & \pi_* (MU) & \pi_* (MU) \\
\downarrow & \downarrow & \downarrow \\
\pi_* (MU_R^{(4)}) & \lambda^{(4)} & \pi_* (MU) \\
\uparrow & \uparrow & \uparrow \\
K & \lambda_1 & \lambda_2 \\
\end{array}
\]

The existence of \( (4) \) is equivalent to that of compatible strict isomorphisms between the formal group laws \( F_j \) classified by the \( j \).
The proof of the Lemma (continued)

Similarly consider the diagram

\[
\begin{array}{cccc}
\pi_{\ast}(MU) & \pi_{\ast}(MU) & \pi_{\ast}(MU) & \pi_{\ast}(MU) \\
\downarrow \lambda_1 & \uparrow \lambda_2 & \downarrow \lambda_3 & \uparrow \lambda_4 \\
\pi_{\ast}(MU^{(4)}_R) & & & \\
\end{array}
\]

The existence of \( \lambda^{(4)} \) is equivalent to that of compatible strict isomorphisms between the formal group laws \( F_j \) classified by the \( \lambda_j \).
Now suppose that $K$ has a $C_8$-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined $C_8$-action on $\text{MU}_{\mathbb{R}}^{(4)}$. 
The proof of the Lemma (continued)

Now suppose that $K$ has a $C_8$-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined $C_8$-action on $\text{MU}_R^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending $x$ to its formal inverse on each of the $F_j$. 
The proof of the Lemma (continued)

Now suppose that $K$ has a $C_8$-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined $C_8$-action on $MU_R^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending $x$ to its formal inverse on each of the $F_j$.

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbb{Z}[[\zeta_8]]$-module structure on each of the $F_j$, which are all isomorphic.
Now suppose that $K$ has a $C_8$-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined $C_8$-action on $MU_4^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending $x$ to its formal inverse on each of the $F_j$.

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbb{Z}[\zeta_8]$-module structure on each of the $F_j$, which are all isomorphic. This proves the first part of the Lemma.
The proof of the Lemma (continued)

For the second part, recall that

\[ D = \Delta_1^{(8)} N_4^8 (\Delta_2^{(4)} N_2^8 (\Delta_4^{(2)}), \]

where

\[ \Delta_g^k = \begin{cases} x_{2^k-1} & \text{for } g = 2 \\ N_4^g (r_{2^k-1}) & \text{otherwise.} \end{cases} \]
The proof of the Lemma (continued)

For the second part, recall that \( D = \Delta_1^{(8)} \mathcal{N}_4^8(\Delta_2^{(4)}) \mathcal{N}_2^8(\Delta_4^{(2)}) \), where

\[
\Delta_k^{(g)} = \begin{cases} 
  x_{2^k-1} & \text{for } g = 2 \\
  \mathcal{N}_4^g(r_{2^k-1}) & \text{otherwise.}
\end{cases}
\]

Since our formal \( A \)-module is 2-typical we can do the calculations using \( BP \) in place of \( MU \).
The proof of the Lemma (continued)

For the second part, recall that $D = \Delta^{(8)}_1 \mathcal{N}^8_4 (\Delta^{(4)}_2) \mathcal{N}^8_2 (\Delta^{(2)}_4)$, where

$$\Delta_k^{(g)} = \begin{cases} x_{2^k - 1} & \text{for } g = 2 \\ \mathcal{N}^g_4 (r_{2^k - 1}) & \text{otherwise.} \end{cases}$$

Since our formal $A$-module is 2-typical we can do the calculations using $BP$ in place of $MU$. Hence we can replace $x_{2^k - 1}$ by $v_k$ and $r_{2^k - 1}$ by $t_k$. 
The proof of the Lemma (continued)

For the second part, recall that $D = \overline{\Delta}^{(8)} N_{4}^{8} (\overline{\Delta}^{(4)} N_{2}^{8} (\overline{\Delta}^{(2)}))$, where

$$\overline{\Delta}^{(g)} \Delta = \begin{cases} x_{2k-1} & \text{for } g = 2 \\ N_{4}^{g} (r_{2k}^{-1}) & \text{otherwise}. \end{cases}$$

Since our formal $A$-module is 2-typical we can do the calculations using $BP$ in place of $MU$. Hence we can replace $x_{2k-1}$ by $v_{k}$ and $r_{2k-1}$ by $t_{k}$. We have $\overline{\Delta}_{k}^{(2)} = v_{k}$. 

The proof of the Lemma (continued)

Using Hazewinkel’s formula we find that
The proof of the Lemma (continued)

Using Hazewinkel’s formula we find that

\[ v_1 \mapsto ( -\pi^3 - 4\pi^2 - 6\pi - 4 ) w = \pi^3 \cdot \text{unit} \cdot w \]
Using Hazewinkel’s formula we find that

\[
\begin{align*}
\nu_1 & \mapsto ( - \pi^3 - 4 \pi^2 - 6 \pi - 4 ) w = \pi^3 \cdot \text{unit} \cdot w \\
\nu_2 & \mapsto ( 4 \pi^3 + 11 \pi^2 + 6 \pi - 6 ) w^3 = \pi^2 \cdot \text{unit} \cdot w^3
\end{align*}
\]
The proof of the Lemma (continued)

Using Hazewinkel’s formula we find that

\[ v_1 \mapsto ( -\pi^3 - 4\pi^2 - 6\pi - 4 ) w = \pi^3 \cdot \text{unit} \cdot w \]

\[ v_2 \mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6) w^3 = \pi^2 \cdot \text{unit} \cdot w^3 \]

\[ v_3 \mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100) w^7 \\
= \pi \cdot \text{unit} \cdot w^7 \]
Using Hazewinkel’s formula we find that

\[ \nu_1 \mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w = \pi^3 \cdot \text{unit} \cdot w \]
\[ \nu_2 \mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 = \pi^2 \cdot \text{unit} \cdot w^3 \]
\[ \nu_3 \mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \]
\[ = \pi \cdot \text{unit} \cdot w^7 \]
\[ \nu_4 \mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15} \]
\[ = \text{unit} \cdot w^7. \]
The proof of the Lemma (continued)

Using Hazewinkel’s formula we find that

\[ v_1 \mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w = \pi^3 \cdot \text{unit} \cdot w \]

\[ v_2 \mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 = \pi^2 \cdot \text{unit} \cdot w^3 \]

\[ v_3 \mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \]
\[ = \pi \cdot \text{unit} \cdot w^7 \]

\[ v_4 \mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15} \]
\[ = \text{unit} \cdot w^7. \]

where each unit is in \( A \).
Using Hazewinkel’s formula we find that

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\[ \nu_2 \mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 = \pi^2 \cdot \text{unit} \cdot w^3 \]

\[ \nu_3 \mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \]

\[ = \pi \cdot \text{unit} \cdot w^7 \]

\[ \nu_4 \mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15} \]

\[ = \text{unit} \cdot w^7. \]

where each unit is in \( A \). It follows that \( \nu_4 \) (but not \( \nu_n \) for \( n < 4 \)) and therefore \( N_2^8(\Delta_4^{(2)}) \) maps to a unit.
The proof of the Lemma (continued)

We have $\Delta_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \rightarrow BP_*^{(4)} \rightarrow R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}. $$
The proof of the Lemma (continued)

We have $\Delta_k^{(2)} = t_k$. We consider the equivariant composite

$$BP^{(2)}_* \to BP^{(4)}_* \to R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2n} - 1}{\pi^n}.$$ 

Using the right unit formula we find that
The proof of the Lemma (continued)

We have \( \Delta_k^{(2)} = t_k \). We consider the equivariant composite

\[
BP_*^{(2)} \rightarrow BP_*^{(4)} \rightarrow R_*
\]

under which

\[
\eta_R(\ell n) \mapsto \frac{\zeta_8^2 w^{2n-1}}{\pi^n}.
\]

Using the right unit formula we find that

\[
t_1 \mapsto (\pi + 2)w = \pi \cdot \text{unit} \cdot w
\]

\[
t_2 \mapsto (\pi^3 + 5\pi^2 + 9\pi + 5)w^3 = \text{unit} \cdot w^3.
\]
The proof of the Lemma (continued)

We have $\Delta_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \rightarrow BP_*^{(4)} \rightarrow R_*$$

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$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2n-1}}{\pi^n}.$$ 

Using the right unit formula we find that

$$t_1 \mapsto (\pi + 2)w = \pi \cdot \text{unit} \cdot w$$
$$t_2 \mapsto (\pi^3 + 5\pi^2 + 9\pi + 5)w^3 = \text{unit} \cdot w^3.$$ 

This means $t_2$ (but not $t_1$) and therefore $N_8(\Delta_2^{(4)})$ maps to a unit.
Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$, 
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The proof of the Lemma (continued)

Finally, we have $\Delta_n^{(8)} = t_n(1) \in BP_*(^4)$, where $t_n(1)$ is the analog of $r_{2^n - 1}(1)$. Then we find

\[
\ell_n(1) \mapsto \frac{w^{2^n - 1}}{\pi^n}
\]

\[
\ell_n(2) \mapsto \frac{(\zeta_8 w)^{2^n - 1}}{\pi^n}.
\]
Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP^{(4)}_*$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$. Then we find

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$$

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$$

This implies

$$
\overline{\Delta}^{(8)}_1 = \ell_1(2) - \ell_1(1) \mapsto w.
$$
The proof of the Lemma (continued)

Finally, we have $\overline{\Delta_n}^{(8)} = t_n(1) \in BP^{(4)}_*$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$. Then we find

$$
\ell_n(1) \mapsto \frac{w^{2^n-1}}{\pi^n},
\ell_n(2) \mapsto \frac{(\zeta_8 w)^{2^n-1}}{\pi^n}.
$$

This implies

$$
\overline{\Delta}^{(8)}_1 = \ell_1(2) - \ell_1(1) \mapsto w.
$$

Thus we have shown that each factor of

$$
D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})
$$

and hence $D$ itself maps to a unit in $R_*$, thus proving the lemma.