



Lin Jinkun's work on the Adams spectral sequence

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Lin's big theorem

Theorem 1 (Lin [Lin03]) *For each prime $p \geq 5$ and each integer $n \geq 0$, the Hopf invariant one element*

$$h_n \in \text{Ext}_A^{1,p^n q}(H^*(K), \mathbf{Z}/(p))$$

is a permanent cycle and therefore represents a map

$$\omega_n : S^{p^n q - 1} \rightarrow K.$$

Here $q = 2p - 2$ and K is Toda's 4-cell complex $V(1)$.

Background

The Adams spectral sequence [Ada58] is a method for computing the graded group of maps $[X, Y]_*$ for suitable spectra X and Y .

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where A is the mod p Steenrod algebra, and the cohomology groups have coefficients in $\mathbf{Z}/(p)$.

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where A is the mod p Steenrod algebra, and the cohomology groups have coefficients in $\mathbf{Z}/(p)$. This group is contravariant in X and covariant in Y .

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and $\text{Ext} := \text{Ext}(\mathbf{Z}/(p)),$
the E_2 -term for $\pi_*(S^0)$

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Ext^1 has basis dual to the set of algebra generators for the mod p Steenrod algebra A , the Bockstein operation Δ and the reduced power operations \mathcal{P}^{p^n} for $n \geq 0$, giving elements

$$a_0 \in \mathrm{Ext}^{1,1} \quad \text{and} \quad h_n \in \mathrm{Ext}^{1,p^n q}.$$

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$$\text{Ext}^{s,t} = \text{Ext}_A^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

Ext^2 has basis dual to the set of algebra relations for the mod p Steenrod algebra A . In particular for each $n > 0$, there is an element $b_{n-1} \in \text{Ext}^{2,p^n q}$ corresponding to the relation

$$\left(\mathcal{P}^{p^{n-1}}\right)^p = \dots$$

Background

$$\begin{aligned} a_0 &\in \text{Ext}^{1,1} & h_n &\in \text{Ext}^{1,p^n q} \\ b_{n-1} &\in \text{Ext}^{2,p^n q} \end{aligned}$$

The following facts about these elements are known.

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- (1) a_0 corresponds to p times the identity map ι and h_0 corresponds to the map $\alpha_1 : S^{q-1} \rightarrow S^0$.

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The following facts about these elements are known.

- (2) Liulevicius [Liu62] (using methods introduced by Adams [Ada60] for $p = 2$) showed that for p odd, h_n for $n > 0$ is not a permanent cycle, but instead there is a nontrivial differential

$$d_2(h_n) = a_0 b_{n-1}.$$

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The following facts about these elements are known.

(3) I showed [Rav78] that for $p \geq 5$ and $n \geq 2$,

$$d_{2p-1}(b_{n-1}) = h_0 b_{n-2}^p \neq 0.$$

Background

$$\begin{aligned} a_0 &\in \text{Ext}^{1,1} & h_n &\in \text{Ext}^{1,p^n q} \\ b_{n-1} &\in \text{Ext}^{2,p^n q} \end{aligned}$$

The following facts about these elements are known.

- (4) Ralph Cohen [Coh81] showed that for $p \geq 3$ and $n \geq 1$, $h_0 b_{n-1}$ is a permanent cycle represented by a map

$$\zeta_{n-1} : S^{N-3} \rightarrow S^0$$

where $N = (p^n + 1)q$. This is the geometric input for Lin's theorem.

The mod p Moore spectrum M

M is known to be a ring spectrum for $p \geq 3$.

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$$S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \xrightarrow{p\iota} S^1.$$

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The element $i_*(a_0)$ vanishes in $\text{Ext}(H^*(M))$, so (2) above is moot here, but we do have

(3') There is a differential

$$d_{2p-1}(i_*(h_n)) = a_1 i_*(b_{n-2})^p$$

where $a_1 \in \text{Ext}^{1,q+1}(H^*(M))$ corresponds to the composite

$$S^q \xrightarrow{i} \Sigma^q M \xrightarrow{\alpha} M$$

and α is the Adams self-map.

The mod p Moore spectrum M

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(4') Cohen [Coh81] showed that for $p \geq 3$ and $n \geq 1$, $i_*(h_0 h_n)$ is a permanent cycle.

Toda's 4-cell complex $K = V(1)$

There is a cofiber sequence

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M$$

and K is known to be a ring spectrum for $p \geq 5$.
Lin's Theorem says that

$$(i'i)_*(h_n) \in \text{Ext}^{1,p^n q}(H^*(K))$$

is a permanent cycle.

Consequences of Lin's theorem

Assuming Lin's theorem is true, let

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denote a map detected by $(i'i)_*(h_n)$.

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There is a well known map (due to Toda)

$$\beta : \Sigma^{(p+1)q} K \rightarrow K$$

whose iterates are all nontrivial.

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Assuming Lin's theorem is true, let

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Lin shows that for $n > 2$ and $0 < s < p^{n-2}$, the composite

$$S^{p^n q - 1} \xrightarrow{\omega_n} K \xrightarrow{\beta^s} \Sigma^{-s(p+1)q} K \xrightarrow{jj'} S^{q+2-s(p+1)q}$$

is nontrivial and is detected by the element

$\gamma_{p^{n-2}/p^{n-2}-s}$ in the Adams-Novikov spectral sequence.

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(3') may imply that for $n > 2$

$$S^{p^n q - 1} \xrightarrow{\omega_n} K \xrightarrow{j'} \Sigma^{q+1} M$$

is detected by

$$i_*(b_{n-2}^p) \in \text{Ext}^{2p, p^n q}(H^*(M))$$

and $j'\omega_n$ may lift to S^{q+1} .

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denote a map detected by $(i'i)_*(h_n)$.
For $n = 2$, the composite

$$S^{p^2 q - 1} \xrightarrow{\omega_2} K \xrightarrow{jj'} S^{q+2}$$

is $\gamma_1 = \alpha_1 \beta_{p-1} \in \pi_{p^2 q - q - 3}(S^0)$.

Consequences of Lin's theorem

Assuming Lin's theorem is true, let

$$\omega_n : S^{p^n q - 1} \rightarrow K$$

denote a map detected by $(i'i)_*(h_n)$.

For $n = 1$, $j'\omega_1 = 0$ and we have

$$\begin{array}{ccccc}
 S^1 & \xleftarrow{\beta_1} & S^{pq-1} & \longrightarrow & \text{pt.} \\
 \uparrow j & & \downarrow \omega_1 & & \downarrow \\
 M & \xrightarrow{i'} & K & \xrightarrow{j'} & \Sigma^{q+1} M
 \end{array}$$

Steps in Lin's argument

Proposition 2 (Toda, [Tod71]) *There is a map*
 $\alpha'' : \Sigma^{q-2}K \rightarrow K$ *with*

$$\begin{array}{ccccccc}
 \Sigma^{q-2}M & \xrightarrow{i'} & \Sigma^{q-2}K & \xrightarrow{\alpha_1 \wedge K} & \Sigma^{-1}K & \xrightarrow{j'} & \Sigma^q M \\
 \downarrow j & & & & & & \downarrow j \\
 S^{q-1} & & & & & & S^{q+1} \\
 \downarrow i & & & \searrow \alpha'' & & & \downarrow i \\
 \Sigma^{q-1}M & \xrightarrow{i'} & \Sigma^{q-1}K & \xrightarrow{\alpha_1 \wedge K} & K & \xrightarrow{j'} & \Sigma^{q+1}M.
 \end{array}$$

Steps in Lin's argument

Theorem 3 (Theorem 3.4 of [Lin03]) *Let $\alpha'' \in \text{Ext}^{1,q-1}(H^*(K), H^*(K))$ detect α'' . Then*

$$(h_0 h_n)'' = \alpha''(i' i)_*(h_n) \in \text{Ext}^{2,N-1}(H^*(K), H^*(K))$$

(where $N = (p^n + 1)q$) is a permanent cycle detecting a map

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The proof of this is very difficult.

Steps in Lin's argument

For the next step we need a minimal Adams resolution

$$\begin{array}{ccccccc} S^0 & \xlongequal{\quad} & E_0 & \xleftarrow{\bar{a}_0} & \Sigma^{-1} E_1 & \xleftarrow{\bar{a}_1} & \Sigma^{-2} E_2 \xleftarrow{\bar{a}_2} \dots \\ & & \downarrow \bar{b}_0 & & \downarrow \bar{b}_1 & & \downarrow \bar{b}_2 \\ H/p & \xlongequal{\quad} & KG_0 & & KG_1 & & KG_2 \end{array}$$

and the cofiber sequence

$$S^0 \xrightarrow{i'i} K \xrightarrow{r} Y \xrightarrow{\epsilon} S^1.$$

Steps in Lin's argument

Corollary 4 (Corollary 3.10 of [Lin03]) *The map η''_n lifts to a map $\eta''_{n,2} : \Sigma^{N-3}K \rightarrow \Sigma^{-2}E_2 \wedge K$, and there is a map $(\eta''_{n,2})_Y$ making the following diagram commute.*

$$\begin{array}{ccccc}
 \Sigma^N K & \xrightarrow{r} & \Sigma^N Y & \xrightarrow{(\eta''_{n,2})_Y} & \Sigma E_2 \wedge K \\
 \downarrow \eta''_{n,2} & & & & \downarrow \bar{b}_2 \wedge K \\
 \Sigma E_2 \wedge K & \xrightarrow{\bar{b}_2 \wedge K} & & & \Sigma K G_2 \wedge K
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 \Sigma E_2 \wedge K & \xrightarrow{\bar{b}_2 \wedge K} & & & \Sigma K G_2 \wedge K
 \end{array}$$

The proof of this is not difficult.

Steps in Lin's argument

The next step concerns the cofiber sequence

$$\Sigma^{q-2} K \xrightarrow{\alpha''} K \xrightarrow{w} X \xrightarrow{u} \Sigma^{q-1} K$$

Lemma 5 (Lemma 4.2 of [Lin03]) *Modulo higher Adams filtration, the composite*

$$\Sigma^{N-3} K \xrightarrow{r} \Sigma^{N-3} Y \xrightarrow{(\eta''_n)_Y} K \xrightarrow{w} X$$

is $\lambda'w(\zeta_{n-1} \wedge K)$ for a nonzero scalar λ' , where ζ_{n-1} is Cohen's map.

Steps in Lin's argument

In other words, there is a map f_1'' making the following diagram commute

$$\begin{array}{ccc}
 \Sigma^N K & \xrightarrow{r} & \Sigma^N Y \\
 \downarrow (f_1'', -\zeta_{n-1} \wedge K) & & \downarrow (\eta_n'')_Y \\
 & & \Sigma^3 K \\
 & & \downarrow w \\
 (\Sigma^{-1} E_4 \wedge X) \vee \Sigma^3 K & \xrightarrow{(\bar{a}_{0,3} \wedge X) \vee w} & \Sigma^3 X
 \end{array}$$

where $\bar{a}_{0,3} = \bar{a}_0 \bar{a}_1 \bar{a}_2 \bar{a}_3$.

Steps in Lin's argument

The proof of this is difficult, as is the derivation of Theorem 1 from it. The latter involves studying a diagram that uses the cofiber sequences

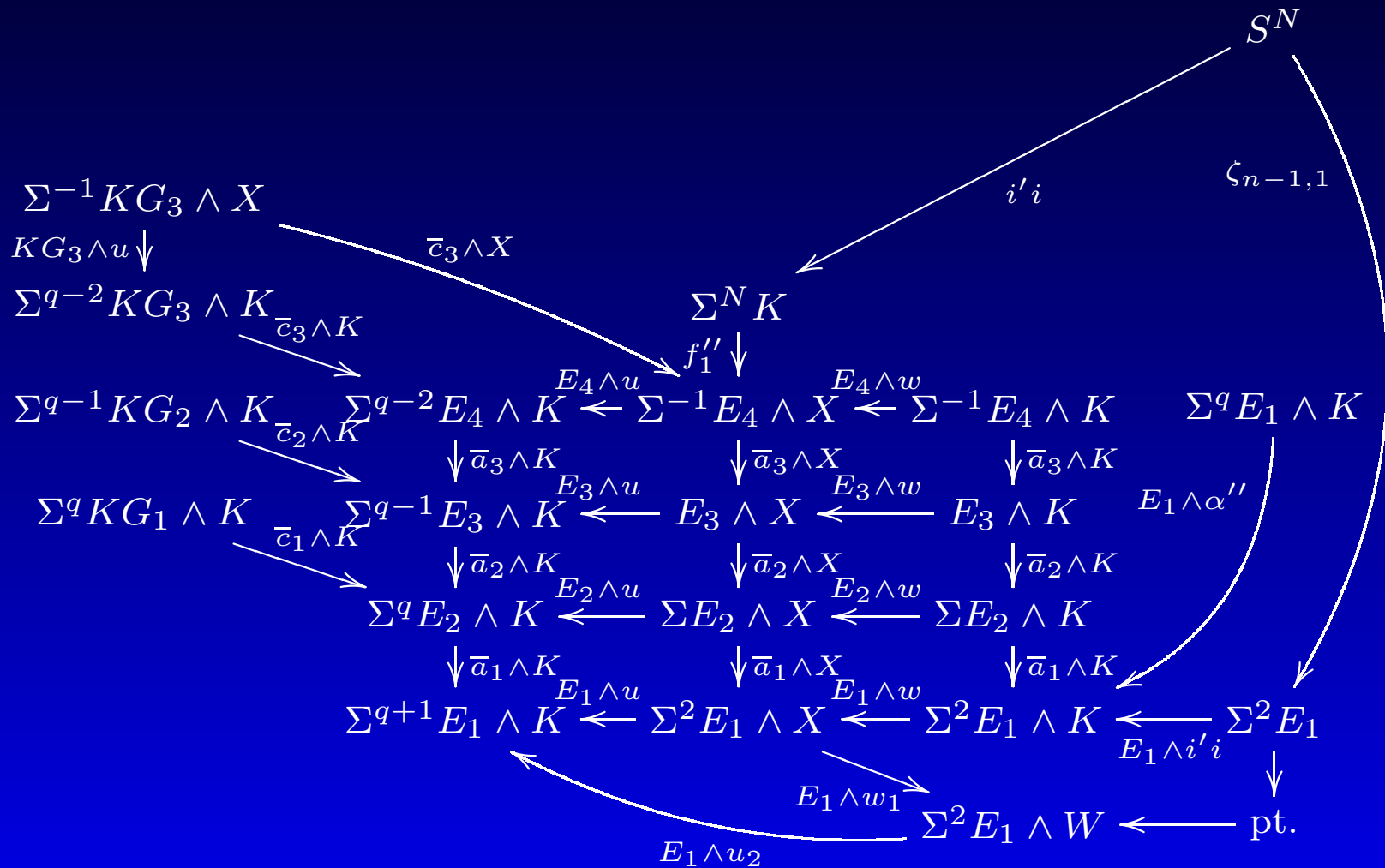
$$\Sigma^{-1}E_{s+1} \xrightarrow{\bar{a}_s} E_s \xrightarrow{\bar{b}_s} KG_s \xrightarrow{\bar{c}_s} E_{s+1},$$

$$S^0 \xrightarrow{wi'i} X \xrightarrow{w_1} W \xrightarrow{u_1} S^1$$

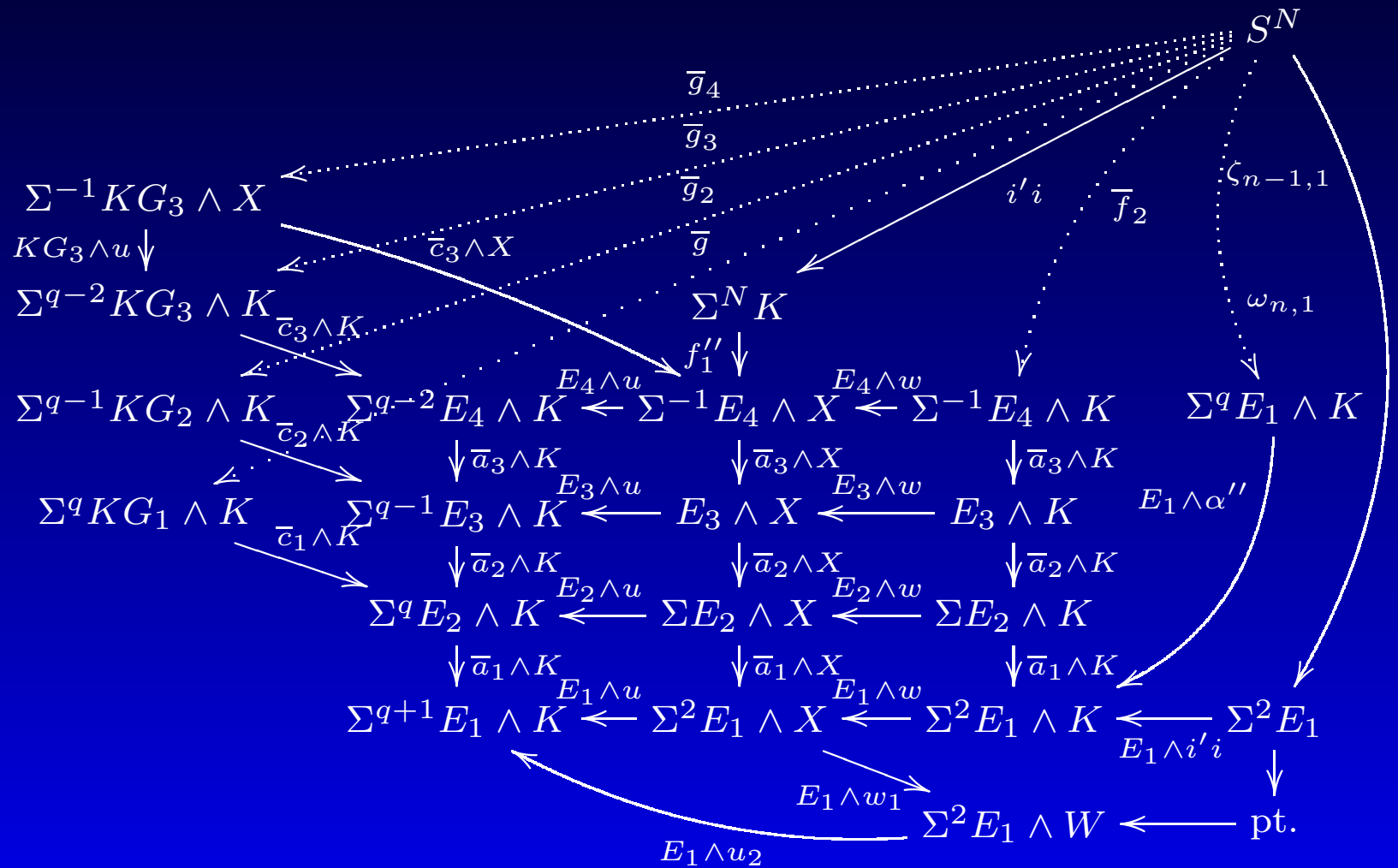
and

$$\Sigma^{q-2}K \xrightarrow{r\alpha''} Y \xrightarrow{w_2} W \xrightarrow{u_2} \Sigma^{q-1}K.$$

Steps in Lin's argument



Steps in Lin's argument



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