An introduction to elliptic cohomology and topological modular forms

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Doug Ravenel

What is elliptic cohomology?

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Equivalently, φ is a homomorphism from the appropriate cobordism ring Ω to R.

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(iii) $F(x,F(y,z)) = F(F(x,y),z)$.

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Let α be a complex line bundle over a space X. It has a Conner-Floyd Chern class $c_1(\alpha) \in MU^2(X)$. Given two such line bundles α_1 and α_2, we have

$$c_1(\alpha_1 \otimes \alpha_2) = G(c_1(\alpha_1), c_1(\alpha_2))$$

where G is the desired formal group law.

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It is also known that the functor

$$X \mapsto MU_*(X) \otimes_\varphi R$$

is a homology theory if φ satisfies certain conditions spelled out in Landweber's Exact Functor Theorem.

What is elliptic cohomology?

Now suppose E is an elliptic curve defined over R. It is a 1-dimensional algebraic group, and choosing a local paramater at the identity leads to a formal group law $\widehat{E}$, the formal completion of E. Thus we can apply the machinery above and get an R-valued genus.

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For example, the *Jacobi quartic*, defined by the equation

$$y^2 = 1 - 2\delta x^2 + \epsilon x^4,$$

is an elliptic curve over the ring

$$R = \mathbf{Z}[1/2, \delta, \epsilon].$$

What is elliptic cohomology?

The resulting formal group law is the power series expansion of

$$F(x,y) = \frac{x\sqrt{1-2\delta y^2+\epsilon y^4}+y\sqrt{1-2\delta x^2+\epsilon x^4}}{1-\epsilon x^2 y^2};$$

this calculation is originally due to Euler. The resulting genus is known to satisfy Landweber's conditions, and this leads to one definition of elliptic cohomology.

What are modular forms?

Recall that an elliptic curve is determined by a lattice in $\mathbf{C}$ generated by 1 and a complex number τ in the upper half plane H. There is an action of the group $SL_2(\mathbf{Z})$ on H given by

$$\tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d} \quad \text{for} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbf{Z}).$$

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What are modular forms?

A *modular form of weight* k is a meromorphic function g defined on the upper half plane satisfying

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k g(\tau).$$

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Here is an example. Let

$$G_k(\tau) := \sum_{m,n\in\mathbf{Z}}{}' \frac{1}{(m\tau + n)^k}$$

where the sum is over all nonzero lattice points. This vanishes if k is odd and is known to converge for $k > 2$.

What are modular forms?

Note that

$$
\begin{aligned}
G_k\left(\frac{a\tau+b}{c\tau+d}\right) \\
&= \sum_{m,n\in\mathbf{Z}}{}' \left(\frac{c\tau+d}{m(a\tau+b)+n(c\tau+d)}\right)^k \\
&= (c\tau+d)^k G_k(\tau).
\end{aligned}
$$

so G_k is a modular form of weight k.

What are modular forms?

Now let $q = e^{2\pi i\tau}$. In terms of it we have

$$G_k(\tau) = 2(2\pi i)^k \frac{B_k}{2k!} \left(1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \right)$$

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and

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

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It is convenient to normalize G_k by defining the Eisenstein series

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It turns out that

$$E_k(\tau) = \sum_{(m,n)=1} \frac{1}{(m\tau + n)^k},$$

where the sum is over pairs of integers that are relatively prime.

What are modular forms?

Let

$$\begin{aligned}\Delta &:= \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} \\ &= q\prod_{n=1}^{\infty}(1-q^n)^{24}\end{aligned}$$

and $\quad j := E_4^3/\Delta.$

Δ is called the discriminant, and the modular function j of weight 0 is a complex analytic isomorphism between $H/SL_2(\mathbf{Z})$ and the Riemann sphere.

What are modular forms?

It is known that the ring of all modular forms with respect to $\Gamma = SL_2(\mathbf{Z})$ is

$$M_*(\Gamma) = \mathbf{C}[E_4, E_6],$$

with (Δ) being the ideal of forms that vanish at $i\infty$, which are called *cusp forms*.

The Weierstrass curve

The *Weierstrass equation* for an elliptic curve in affine form is

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

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It is known that any elliptic curve is isomorphic to one of this form.

The Weierstrass curve

The Eisenstein series are related to the a_k by

$$E_4 = b_2^2 - 24b_4$$

and

$$E_6 = -b_2^3 + 36b_2b_4 - 216b_6$$

where

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 \\ b_4 &= a_1a_3 + 2a_4 \\ b_6 &= a_3^2 + 4a_6. \end{aligned}$$

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This means there is a formal group law defined over the ring

$$E\ell\ell_* = \mathbf{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}]$$

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$$E\ell\ell_* = \mathbf{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}]$$

and the resulting genus is Landweber exact. Thus we get a spectrum $E\ell\ell$ with $\pi_*(E\ell\ell) = E\ell\ell_*$.

The Weierstrass curve

Consider the affine coordinate change

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It can be thought of as an action of an affine goup G of 3×3 matrices given by

$$\begin{bmatrix} 1 & s & t \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + sy + t \\ y + r \\ 1 \end{bmatrix}$$

The Weierstrass curve

Under it we get

$$
\begin{aligned}
a_6 &\mapsto a_6 + a_4\,r + a_3\,t + a_2\,r^2 \\
&\qquad +a_1\,r\,t + t^2 - r^3 \\
a_4 &\mapsto a_4 + a_3\,s + 2\,a_2\,r \\
&\qquad +a_1(r\,s + t) + 2\,s\,t - 3\,r^2 \\
a_3 &\mapsto a_3 + a_1\,r + 2\,t \\
a_2 &\mapsto a_2 + a_1\,s - 3\,r + s^2 \\
a_1 &\mapsto a_1 + 2\,s.
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The spectrum tmf

Very roughly speaking, there is an action of the group G on the spectrum $E\ell\ell$ inducing the above action on its homotopy groups.

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The homotopy fixed point set of this action is tmf.

There is a spectral sequence converging to $\pi_*(tmf)$ with

$$E_2 = H^*(G; E\ell\ell_*).$$

The modular forms E_4 and E_6 are both invariant under this action, and

$$H^0(G; E\ell\ell_*) = \mathbf{Z}[E_4, E_6][\Delta^{-1}]$$

The spectrum tmf

The coordinate change above can be used to define a Hopf algrebroid (A, Γ) with

$$\begin{aligned} A &= \mathbf{Z}[a_1, a_2, a_3, a_4, a_6] \\ \Gamma &= A[r, s, t] \end{aligned}$$

and right unit $\eta_R : A \to \Gamma$ given by the formulas above.

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and right unit $\eta_R : A \longrightarrow \Gamma$ given by the formulas above. It was first described by Hopkins and Mahowald in *From elliptic curves to homotopy theory*. Its Ext group is the cohomology group mentioned above. Tilman Bauer has written a nice account of this calculation.