Toward higher chromatic analogs of elliptic cohomology and topological modular forms

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- L_0 is rationalization. Rational stable homotopy theory is very well understood. It detects only the 0-stem in the stable homotopy groups of spheres.

- L_1 is localization with respect to K-theory. It detects the image of J and the α family in the stable homotopy groups of spheres. The Lichtenbaum-Quillen conjecture is a statement about L_1 of algebraic K-theory.

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The stable homotopy category localized a prime p can be studied via a series of increasingly complicated localization functors L_n for $n \geq 0$, which detect “v_n-periodic” phenomena.

- L_2 is equivalent to localization with respect to elliptic cohomology as defined above. It detects the β family in the stable homotopy groups of spheres. Davis’ nonimmersion theorem for real projective spaces was proved using related methods. The theory of topological modular forms of Hopkins *et al* is a refinement of elliptic cohomology.

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The stable homotopy category localized a prime p can be studied via a series of increasingly complicated localization functors L_n for $n \geq 0$, which detect "v_n-periodic" phenomena.

- For $n > 2$ there is no comparable geometric definition of L_n, which can only be constructed by less illuminating algebraic methods related to BP-theory. It detects higher Greek letter families in the stable homotopy groups of spheres. The nth Morava K-theory is closely related to it.

The height of a formal group law

Definition 1 *Let F be 1-dimensional formal group law. For a positive integer m, the m-series is defined inductively by*

$$[m]_F(x) = F(x, [m-1]_F(x))$$

where $[1]_F(x) = x$.

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for some nonzero $a \in k$. The **height** *of F is the integer n. If $[p]_F(x) = 0$ (which happens when $F(x, y) = x + y$), the height is defined to be ∞.*

Examples of m-series

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$$\begin{aligned} [m](x) &= (1+x)^m - 1 \\ &= mx + \binom{m}{2} x^2 + \dots . \end{aligned}$$

- For $F(x,y) = \frac{x+y}{1+xy}$, we have

$$\begin{aligned} [m](x) &= \sum_i \binom{m}{2i+1} x^{2i+1} \Big/ \sum_i \binom{m}{2i} x^{2i} \\ &= \frac{mx + \binom{m}{3} x^3 + \dots}{1 + \binom{m}{2} x^2 + \dots} \end{aligned}$$

Examples of heights

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- The formal group law associated with an elliptic curve is known to have height at most 2.
- v_n-periodic phenomena (the nth layer in the chroatic tower) are related to formal group laws of height n.

Question

How can we attach formal group laws of height > 2 to geometric objects (such as algebraic curves) and use them get insight into cohomology theories that go deeper into the chromatic tower?

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- Let C be a curve of genus g over some ring R.
- Its Jacobian $J(C)$ is an abelian variety of dimension g.
- $J(C)$ has a formal completion $\widehat{J}(C)$ which is a g-dimensional formal group law.
- If $\widehat{J}(C)$ has a 1-dimensional summand, then Quillen's theorem gives us a genus (in the sense of the talk or 2/27) associated with the curve C.

Caveat

Note that a 1-dimensional summand of the formal completion $\widehat{J}(C)$ is *not* the same thing as 1-dimensional factor of the Jacobian $J(C)$. The latter would be an elliptic curve, whose formal completion can have height at most 2. There is a theorem that says if an abelian variety A has a 1-dimensional formal summand of height n for $n > 2$, then the dimension of A (and the genus of the curve, if A is a Jacobian) is at least n.

Artin-Schreier curves

Theorem 2 *Let* $C(p, f)$ *be the curve over* $\mathbf{F}_p$ *defined by the affine equation*

$$y^e = x^p - x \qquad \textit{where } e = p^f - 1.$$

(Assume that $f > 1$ *when* $p = 2$*.) Then its Jacobian has a 1-dimensional formal summand of height* $(p-1)f$.

The resulting genus is *not* Landweber exact, so this does not lead to a cohomology theory.

The Lubin-Tate lifting

Theorem 3 *Let $C'(p,f)$ be the curve over over the ring $E_* = \mathbf{Z}_p[[u_1,\ldots,u_{h-1}]][u,u^{-1}]$ defined by*

$$y^e = x^p - u^m x + \sum_{i=0}^{h-2} u_{i+1} x^{p-1-[i/f]} y^{p^{f-1}-p^{i-[i/f]f}}$$

where $m = (p-1)e$.

Then its Jacobian has a formal 1-dimensional subgroup isomorphic to the Lubin-Tate [LT65] lifting of the formal group law above. The resulting genus satisfies Landweber's exactness criteria, so we get a cohomology theory.

My dream curve

Conjecture 4 *Let $C''(p,f)$ be the curve over over the ring*

$$R = \mathbf{Z}_p[u, u^{-1}][[a_{(p-s)e-qt} : s, t \geq 0,\ es + pt < pe]]$$

defined by

$$y^e \;=\; x^p - u^m x + \sum_{s,t} a_{(p-s)e-pt} x^s y^t$$

where $|u| = 2$ and $|a_i| = 2i$.

Then its Jacobian has a formal 1-dimensional subgroup, subject to certain divisibility conditions among the a_i for $f > 2$. The resulting genus also satisfies Landweber's exactness criteria.

My dream curve

This last curve is amenable to change of coordinates and possibly to a calculation generalizing that of Hopkins-Mahowald for tmf.

Properties of $C(p, f)$

Recall that $C(p, f)$ is the Artin-Schreier curve over $\mathbf{F}_p$ defined by the affine equation

$$y^e = x^p - x, \qquad \text{where } e = p^f - 1,$$

and its Jacobian has a 1-dimensional formal summand of height $h = (p-1)f$.

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and its Jacobian has a 1-dimensional formal summand of height $h = (p-1)f$.

- Its genus is $(p-1)(e-1)/2$. (Thus it is zero in the excluded case $(p, f) = (2, 1)$.)

Properties of $C(p, f)$

- It has an action by the group

$$G = \mathbf{F}_p \rtimes \mu_m \qquad \text{where } m = (p-1)e$$

and μ_m is the group of mth roots of unity, given by

$$(x, y) \mapsto (\zeta^e x + a, \zeta y)$$

for $a \in \mathbf{F}_p$ and $\zeta \in \mu_m$. This group is a maximal finite subgroup of the hth Morava stabilizer group, and it acts appropriately on the 1-dimensional formal summand.

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- For $f = 1$ (and $p > 2$) Theorem 3 was proved by Gorbunov-Mahowald [GM00].

Examples of these curves

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- $C(2,3)$ has genus 3 and a 1-dimensional formal summand of height 3.
- $C(2,4)$ and $C(3,2)$ each have genus 7 and a 1-dimensional formal summand of height 4.

Remarks

- Theorem 2 was known to and cited by Manin in 1963 [Man63]. Most of what is needed for the proof can be found in Katz's 1979 Bombay Colloquium paper [Kat81] and in Koblitz' Hanoi notes [Kob80].

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- The original proof rests on the determination of the zeta function of the curve by Davenport-Hasse in 1934 [HD34], and on some properties of Gauss sums proved by Stickelberger in 1890 [Sti90]. The method leads to complete determination of $\widehat{J}(C(p,f))$.

Remarks

- We have reproved Theorem 2 using Honda's theory of commutative formal group laws developed in the early '70s. This proof does not rely on knowledge of the zeta function and can be modified to prove Theorem 3 and presumably Conjecture 4.

Notation for Honda Theory

- Given a power series $f(x_1, x_2, \ldots)$ in several variables over $\mathbf{Z}_p$ or $\mathbf{Q}_p$, let $T^n(f)$ be the power series obtained from f by replacing each variable by its p^nth power. This leads to an action of the ring $\mathbf{Z}_p[T]$ on the power series ring $R = \mathbf{Q}_p[[x_1, x_2, \ldots]]$. Similarly a vector of d such power series admits an action by the matrix ring $M_d(\mathbf{Z}_p[T])$.

Notation for Honda Theory

- Suppose we have a d-dimensional formal group law F over $\mathbf{Z}_p$. F is characterized by its logarithm f, which is a vector of d power series in d variables over the field $\mathbf{Q}_p$. Given a matrix $H = \sum_i C_i T^i$ with $C_i \in M_d(\mathbf{Z}_p)$, define

$$(H * f)(x_1, \ldots, x_d) = \sum_i C_i f(x_1^{p^i}, \ldots, x_d^{p^i}).$$

Honda theory

Definition 5 *We say that* H *is a* **Honda matrix** *for* F *(or for the vector* f*) and that* F is of type H, *if* $H \equiv pI_d$ *modulo* T *(*I_d *is the* $d \times d$ *identity matrix) and* $(H * f)(x) \equiv 0$ *modulo* (p)*. Two such matrices are said to be* **equivalent** *if they differ by unit multiplication on the left.*

Honda theory

Theorem 6 (Honda, 1970 [Hon70]) *The strict isomorphism classes of d-dimensional formal group laws over $\mathbf{Z}_p$ correspond bijectively to the equivalence classes of matrices*

$$H \in M_d(\mathbf{Z}_p)\langle\langle T\rangle\rangle$$

congruent to pI_d modulo degree 1. H and f are related by the formula

$$f(x) = (H^{-1} * p)(x).$$

Examples of Honda matrices

- For $d = 1$, let H be the 1×1 matrix with entry $h = p - T^n$ for a positive integer n. Then

$$f(x) = \sum_{i \geq 0} \frac{x^{p^{ni}}}{p^i}$$

and F is the formal group law for the Morava K-theory $K(n)_*$.

Examples of Honda matrices

- Let $A = \mathbf{Z}_p[[u_1, u_2, \ldots u_{n-1}]]$ for a positive integer m, and let $Tu_i = u_i^p$. Let H be the 1×1 matrix with entry

$$h = p - T^n - \sum_{0<i<n} u_i T^i.$$

  Then $f(x)$ is the logarithm for the Lubin-Tate lifting of the formal group law above.

More Honda theory

Question: How can we find the Honda matrix for the formal completion of the Jacobian of an algebraic curve?

More Honda theory

Theorem 7 (Honda, 1973 [Hon73]) *Let C be a curve of genus g over $\mathbf{Z}_p$ with smooth reduction modulo p, let*

$$\{\omega_1, \ldots, \omega_g\}$$

be a basis for the space of holomorphic 1-forms of C written as power series in a local parameter y, and let

$$\psi_i = \int_0^y \omega_i.$$

Then if H is a Honda matrix for the vector $(\psi_1, \ldots, \psi_g)$, it is also one for $\widehat{J}(C)$, the formal completion of the Jacobian $J(C)$.

More Honda theory

Note that ψ above is a vector of power series in one variable over $\mathbf{Q}_p$, while the logarithm of $\widehat{J}(C)$ is a vector of power series in g variables. The theorem asserts that they have the same Honda matrix.

Honda theory for $C(p, f)$

Recall that our curve $C(p, f)$ is defined by the affine equation

$$y^e = x^p - x \qquad \text{where } e = p^f - 1.$$

Its genus is $g = (e-1)(p-1)/2$. A basis for the holomorphic 1-forms for $C(p, f)$ is

$$\{\omega_{j,k} \colon j, k \geq 0,\ ej + pk < 2g - 1\},$$

where

$$\omega_{j,k} = \frac{x^j y^k \mathrm{d}y}{1 - px^{p-1}}.$$

Honda theory for $C(p, f)$

We denote the integral of its expansion in terms of y by ψ_{ej+k+1}, and we have

$$\psi_{ej+k+1} = \sum_{i \geq 0} \binom{pi+j}{i} \frac{y^{mi+ej+k+1}}{mi+ej+k+1}.$$

This enables us to prove Theorem 2.

The Honda matrix for $C(2,3)$

For $C(2,3)$ (where $g=3$ and $m=7$), the integrals have the form

$$\begin{aligned} \psi_1 &\in y\mathbf{Q}_2[[y^7]] \\ \psi_2 &\in y^2\mathbf{Q}_2[[y^7]] \\ \psi_3 &\in y^3\mathbf{Q}_2[[y^7]] \end{aligned}$$

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More explicitly

$$\psi_k = \sum_{i \geq 0} \binom{2i}{i} \frac{y^{7i+k}}{7i+k}.$$

The Honda matrix for $C(2,3)$

This means that

$$\begin{array}{rcll} T\psi_1 & \in & y^2\mathbf{Q}_2[[y^7]] & \\ T\psi_2 & \in & y^4\mathbf{Q}_2[[y^7]] & \\ T\psi_3 & \in & y^6\mathbf{Q}_2[[y^7]] & \\ T^2\psi_1 & \in & y^4\mathbf{Q}_2[[y^7]] & \\ T^2\psi_2 & \in & y^8\mathbf{Q}_2[[y^7]] & \subset y\mathbf{Q}_2[[y^7]] \\ T^2\psi_3 & \in & y^{12}\mathbf{Q}_2[[y^7]] & \subset y^5\mathbf{Q}_2[[y^7]] \end{array}$$

The Honda matrix for $C(2,3)$

This implies that the Honda matrix has the form

$$H = \begin{bmatrix} h_{1,1}(T^3) & T^2 h_{1,2}(T^3) & 0 \\ T h_{2,1}(T^3) & h_{2,2}(T^3) & 0 \\ 0 & 0 & h_{3,3}(T^3) \end{bmatrix}$$

where

$$h_{i,j}(T^3) = \sum_{k \geq 0} h_{i,j,k} T^{3k}$$

with $h_{i,j,k} \in \mathbf{Z}_{(2)}$ and $h_{i,i,0} = 2$.

The Honda matrix for $C(2,3)$

This means that the 3-dimensional formal group law has a 1-dimensional summand. Since

$$\begin{aligned}\psi_3 &= \frac{y^3}{3} + \frac{y^{10}}{5} + \frac{6y^{17}}{17} + \frac{5y^{24}}{6} + \dots \\ &\equiv y^3 + y^{10} + \frac{y^{24}}{2} + \dots \qquad \bmod 2,\end{aligned}$$

$h_{3,3}$ is roughly $2 - T^3$, and the 1-dimensional summand has height 3.

The Honda matrix for $C(3,2)$

For $C(3,2)$ (where $g = 7$ and $m = 16$), we get integrals ψ_k for

$$k \in S = \{1,\ 2,\ 3,\ 4,\ 5,\ 9,\ 10\}\,.$$

Explicitly,

$$\psi_k = \sum_{i \geq 0} \binom{3i + [k/8]}{i} \frac{y^{16i+k}}{16i+k}.$$

The Honda matrix for $C(3,2)$

A similar computation shows that ψ_5 corresponds to a 1-dimensional formal summand. The argument boils down to seeing how the orbits O of $\mathbf{Z}/(16)$ under multiplication by 3 intersect the set S above. Each such intersection corresponds to a formal summand whose dimension is the cardinality of $O \cap S$ and whose height is the cardinality of O. One such orbit is $\{5, 15, 13, 7\}$, whose intersection with S is the singleton $\{5\}$.

The Honda matrix for $C(3,2)$

We find that

$$\psi_5 \equiv -y^5 + y^{21} - y^{117} + y^{261} + \frac{2\,y^{405}}{3} + \dots \qquad \mod 3,$$

which leads to a Honda eigenvalue of roughly $3 - T^4$, so the height of the 1-dimensional formal summand is 4 as claimed.

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