MATH 162
Midterm 2 ANSWERS
November 18, 2005

1. (10 points) Does the following integral converge or diverge? To get full credit, you must justify your answer.
\[ \int_1^\infty \frac{3x - 2}{x^3 + 4x^2 + 2x + 4} \, dx \]

You may not be able to determine the area is exactly, but you can still be sure of your answer to this question.

Answer:
The integral converges. We concentrate on the leading terms in the numerator and denominator. Notice that for \( x \geq 1 \)

\[
3x - 2 \leq 3x \\
x^3 + 4x^2 + 2x + 4 \geq x^3.
\]

and so

\[
\int_1^\infty \frac{3x - 2}{x^3 + 4x^2 + 2x + 4} \, dx \leq \int_1^\infty \frac{3x}{x^3} \, dx \\
= \int_1^\infty \frac{3}{x^2} \, dx \\
= -3 \frac{1}{x} \bigg|_1^\infty \\
= 3
\]

Warning! Just because \( f(x) \to 0 \) doesn’t mean that \( \int_1^\infty f(x) \, dx \) converges. Consider \( \int_1^\infty (1/x) \, dx \).

2. (10 points) Find the length of the curve
\[ y = \frac{x^3}{6} + \frac{1}{2x} \]

from \( x = 1 \) to \( x = 2 \).
Answer:

First note that
\[ \frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}. \]

We have length = \( \sum \Delta L = \sum \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum \sqrt{1 + (\Delta y/\Delta x)^2} \Delta x \)

Taking the limit we obtain the corresponding integral:
\[
L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \\
= \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} \, dx \\
= \int_1^2 \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} \, dx \\
= \int_1^2 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} \, dx \\
= \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2 \, dx \\
= \frac{x^3}{6} \bigg|_1^2 - \frac{1}{2x} \bigg|_1^1 \\
= \frac{8}{6} - \frac{1}{6} - \frac{1}{4} + \frac{1}{2} \\
= \frac{11}{12}
\]

3. (10 points) Suppose one were to roll a disk of radius 1 inside of a larger circle of radius 2. The small circle starts at the bottom, and rolls in the counterclockwise direction until it has covered a distance of \(2\pi\) along the circumference of the large circle. Would the disk make more or less than a complete revolution, or exactly equal? **Warning:** To get full credit, you must justify your answer.
Answer:

The answer is always “less than one revolution” no matter how big the outer circle. It is exactly equal to one revolution if the outer “circle” is a straight line (cycloid).

Suppose we keep track of a point which is initially at the bottom of the small circle inside a very big circle. The point is initially in contact with the circumference of the large circle. Since the small circle has radius $2\pi$, the point would again come into contact with the circumference of the large circle when a distance of $2\pi$ has been covered on the circumference of the large circle. But when that happens, the point will not be at the very bottom of the small circle, but it will be to the right of the bottom. Thus, the small circle has made less than a complete revolution.

If you decrease the size of the big circle so the ratio is two to one, then the first time the point of the small circle will touch the big circle will be when the small circle has gone around half way inside the big circle and the small circle will have rotated one half revolution.

4. (10 points) Write an integral which would represent the area of the surface obtained by rotating the curve $y = e^x$, $0 \leq x \leq 1$ about the $y$-axis. Your answer should involve an integral of just one variable, without any derivatives in your formula.

Answer:

The surface area is approximately: \[ \sum 2\pi r \sqrt{(\Delta x)^2 + (\Delta y)^2} \] We can change this to represent a Riemann sum either in terms of $x$ or in terms of $y$. To simplify this in terms of $y$ we factor out $\Delta y$ and use $r = x = \ln y$ and then surface area = \[ \sum 2\pi \ln y \sqrt{(\Delta x / \Delta y)^2 + 1} \Delta y \] In terms of $y$, the limits of integration should be $e^0 = 1$ and $e^1 = e$. Next, we must express $x$ in terms
of $y$, and take its derivative, to obtain

$$x = \ln y$$

$$\frac{dx}{dy} = \frac{1}{y}$$

Then, the formula for surface area would be

$$A = \int_1^e 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$= \int_1^e 2\pi \ln y \sqrt{1 + \frac{1}{y^2}} \, dy$$

5. (10 points) Sketch the following parametric curve for $-\infty < t < \infty$, $t \neq 0$.

$$x = \frac{1}{t}$$

$$y = \cos t$$

Answer:

As $t$ goes to infinity $1/t$ goes to zero, hence $\cos(1/t)$ goes to 1, so the graph will have a horizontal asymptote line at $y = 1$. The horizontal asymptote as $t$ goes to negative infinity will also give a horizontal asymptote of $y = 1$.

Near $t = 0$, $t > 0$ $1/t$ goes to plus infinity so $\cos(1/t)$ oscillates infinitely often as $x = 1/t$ approaches 0. It’s hard to graph this accurately of course or draw it, since the wiggle rate goes to infinity. Here’s an attempt:

6. (10 points) Find the derivative $dy/dx$ of the following parametric curve as a function
of \( t \).

\[
x = e^{2t} \\
y = \cos t
\]

For which values of \( t \) is the tangent line to the curve horizontal?

**Answer:**

Using the formula

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt}
\]

we get

\[
\frac{dy}{dx} = \frac{-\sin t}{2e^{2t}}
\]

The slope would be horizontal where \( \sin t = 0 \), in other words where \( t \) is a multiple of \( \pi \).

7. (10 points) Sketch the following polar curve.

\[
r = \frac{1}{\sin 2\theta}
\]

**Answer:**

The graph looks like \( \text{graph} \).

We see that \( \sin(2\theta) \) goes to zero as \( \theta \) approaches 0, \( \pi/2 \), \( \pi \) and \( 2\pi \) which creates those portions where the graph is unbounded. Note that in the region when \( \pi/2 < \theta < \pi \) \( r \) is actually negative and traces out the graph in the 4th quadrant.

In the direction \( \pi/4 \), \( \sin(2\theta) \) reaches its maximum value of one and the graph comes closest to the origin. This repeats in the other quadrants.

8. (10 points) Find the limit of the following sequences:

(a) \( a_n = \arcsin \left( \frac{n-1}{n} \right) \) \quad \lim_{n \to \infty} a_n = \text{ANSWER: }
(b) \( b_n = \sqrt{n^2 - 6n + 8} - n \) 

\[ \lim_{n \to \infty} b_n = \text{ANSWER: } \]

\textbf{Answer:}

Since \( \arcsin \) is continuous, \( \lim_{n \to \infty} \arcsin\left(\frac{n-1}{n}\right) = \arcsin(\lim_{n \to \infty}) = \arcsin(1) = \pi/2 \).

To see the second limit more clearly multiply by the conjugate (to “rationalize the numerator”).

\[ b_n = \left(\sqrt{n^2 - 6n + 8} - n\right) \frac{\sqrt{n^2 - 6n + 8} + n}{\sqrt{n^2 - 6n + 8} + n} = \frac{-6n + 8}{\sqrt{n^2 - 6n + 8} + n} \]

and now by comparing the terms which grow fastest it’s easy to see that the limit is \(-3\).

\textbf{9. (10 points)} Express the following continued fraction as a rational number.

\[ 23.4343434 \ldots \]

\textbf{Answer:}

\[ 23.4343434 \ldots = 20 + \frac{1}{10} \times 34.343434 \ldots \]

\[ = 20 + \frac{1}{10} \times 34 \times 1.010101 \ldots \]

\[ = 20 + \frac{1}{10} \times 34 \times \left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \ldots\right) \]

\[ = 20 + \frac{1}{10} \times 34 \times \frac{1}{1 - \frac{1}{100}} \]

\[ = 20 + \frac{1}{10} \times 34 \times \frac{100}{99} \]

\[ = 20 + \frac{340}{99} \]

\[ = \frac{2320}{99} \]
There are other ways to solve it. For example,

\[
23.434343 \ldots = 23 + \frac{1}{100} \times 43.434343 \ldots \\
= 20 + \frac{43}{100} \times \frac{1}{1 - \frac{1}{100}} \\
= 20 + \frac{43}{99} \\
= \frac{2320}{99}
\]

10. (10 points)

What kind of figures are represented by \( r = ae^{\theta+b} \)

Answer:

These are spirals, actually called logarithmic spirals.

Sketch a graph when \( a = 1 \) and \( b = 0 \)

Answer:

This is not to scale, in order to show this I had actually plot \( r = e^{1t} \) to be able to see a reasonable amount of the plot. The plot only shows the part with \( t > 0 \). For \( t < 0 \) the plot spirals into the origin, but doesn’t reach it.

How does this graph change when \( b = 2\pi \) instead of 0?

Answer:

This has the effect of rotating the graph clockwise by one revolution, the \( \theta = 0 \) intersection
with the $x$ axis will slowly move outward. After a complete revolution the $\theta = 0$ intersection will have moved out from 1 to $e^{2\pi}$. In fact since $e^{\theta + 2\pi} = e^\theta e^{2\pi}$ the effect of a rotation is exactly the same as the effect of an expansion for logarithmic spirals. Not for other curves however.

If you draw the entire spiral ($-\infty < t < \infty$) then when you rotate the spiral once or expand it by $e^{2\pi}$ the new graph sits right on top of the old one, so nothing changes.

How does this graph change when $b = \pi$?

**Answer:**

The graph will have rotated $\pi$ clockwise or expanded by $e^{\pi}$. Either interpretation is possible.

How does increasing $a$ change the graph?

**Answer:**

Increasing $a$ expands the size and spacing of the spiral by a factor $a$. 