The first part of this lecture is about Taylor’s Inequality (page page 737/763 of the 6th/5th edition of Stewart), which is the main tool for estimating the accuracy of a power series approximation to your favorite function.

The Taylor series for \( f(x) \) at \( a \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \cdots,
\]

the \( n \)th Taylor polynomial is

\[
T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!},
\]

and the \( n \)th Taylor remainder is

\[
R_n(x) = f(x) - T_n(x).
\]

If we know that \( R_n(x) \) is small, we know that \( T_n(x) \) is a good approximation for \( f(x) \).

Here is what Taylor’s Inequality says. Assume that \( |x - a| \leq d \). Suppose we know that

\[
|f^{(n+1)}(x)| \leq M
\]

for all such \( x \). Then

\[
|R_n(x)| \leq \frac{M|x - a|^{n+1}}{(n + 1)!}
\]

for all such \( x \).
WHY IS TAYLOR’S INEQUALITY TRUE?

The simplest case is \( n = 0 \). We have \( T_0(x) = f(a) \) so \( R_0(x) = f(x) - f(a) \).

Hence (1) says that if \( |f'(x)| < M \), then

\[
|f(x) - f(a)| = |R_0(x)| \leq M|x - a|
\]

so

\[
\left| \frac{f(x) - f(a)}{x - a} \right| \leq M.
\]

The fraction on the left is the subject of the Mean Value Theorem on page 282/291. It says there is a number \( c \) between \( x \) and \( a \) with

\[
f(x) - f(a) = f'(c)(x - a).
\]

We are assuming that \( |f'(c)| \leq M \) (whatever \( c \) is), so (1) holds for \( n = 0 \).

Now consider the case \( n = 1 \). We are assuming that

\[-M \leq f''(x) \leq M,
\]

For \( x > a \) this means that

\[-\int_a^x Mt \, dt \leq \int_a^x f''(t) \, dt \leq \int_a^x Mt \, dt.
\]

Each of these integrals is easy to calculate, and we get

\[-M(x - a) \leq f'(x) - f'(a) \leq M(x - a).
\]

Now we integrate again and get

\[-M \int_a^x (t - a) \, dt \leq \int_a^x [f'(t) - f'(a)] \, dt \leq M \int_a^x (t - a) \, dt.
\]

These integrals are also easy to work out, and we get

\[-\frac{M(x - a)^2}{2} \leq f(x) - f(a) - f'(a)(x - a) \leq \frac{M(x - a)^2}{2}.
\]

The middle term is \( R_1(x) \), so we have

\[
|R_1(x)| \leq \frac{M|x - a|^2}{2},
\]

which is Taylor’s Inequality for \( n = 1 \). For \( x < a \), we can make a similar calculation integrating from \( x \) to \( a \) instead of from \( a \) to \( x \).

What about larger values of \( n \)? We can proceed in the same way. Starting with

\[-M \leq f^{(n)}(x) \leq M,
\]

we integrate \( n + 1 \) times and end up with

\[-\frac{M(x - a)^{n+1}}{(n + 1)!} \leq R_n(x) \leq \frac{M(x - a)^{n+1}}{(n + 1)!},
\]

which means

\[
|R_n(x)| \leq \frac{M|x - a|^{n+1}}{(n + 1)!},
\]

as advertised.
**The binomial theorem**

One can work out the Maclaurin series for \((1 + x)^k\) for your favorite real number \(k\). This is done pages 741-742/772-773. It is

\[
\sum_{n=0}^{\infty} \frac{x^n k(k-1)(k-2) \cdots (k-n+1)}{n!} = 1 + kx + \cdots
\]

The coefficient of \(x^n\) is an expression that appears so often in mathematics, that there is a special notation for it,

\[
\binom{k}{n} = \frac{k(k-1)(k-2) \cdots (k-n+1)}{n!},
\]

which reads “\(k\) choose \(n\)” and is known as a binomial coefficient. The reason for the word “choose” is that when \(k\) is a positive whole number, \(\binom{k}{n}\) is the number of ways to choose a set of \(n\) objects from a set of \(k\) objects. For example, if you wanted to 5 people for a basketball team out of a group of 100, the number of way you could do it would be

\[
\binom{100}{5} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}{5!} = 75287520.
\]

The ratio test shows that (2) converges for \(|x| < 1\). If \(k\) happens to be a positive whole number, then all terms beyond the \(k\)th one are zero, and we get

\[
(1 + x)^k = 1 + kx + \cdots + kx^{k-1} + x^k
\]

They should have told you this in high school, and it converges for all \(x\) since it is a polynomial.

We will be interested in the case \(k = -1/2\), for which we have

\[
\frac{1}{\sqrt{1+x}} = (1 + x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n = 1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots
\]
Einstein’s equation

In Einstein’s Theory of Special Relativity, first published in 1905, there is a
formula concerning the mass of a moving object. It says that if an object with
mass $M$ is moving with velocity $v$ it gets heavier, and the new mass is

$$M' = \frac{M}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where $c$ denotes the speed of light.

The ratio $v/c$ is usually small. No human being has ever travelled at a speed of
$c/10,000$, and only astronauts and fighter pilots have gone faster than $c/1,000,000$.
This means that the quantity

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

is usually only slightly larger than 1, so close than nobody would notice the dif-
ference between $M'$ and $M$. On the other hand as the velocity $v$ approaches the
speed of light, $\beta$ approaches infinity. This is part of the reason that it is impossible
for anything to move faster than the speed of light.

Using the binomial theorem (with $k = -1/2$ and $x = -v^2/c^2$), we can get a
series for the ratio $\beta$, namely

$$\beta = 1 + \frac{v^2}{2c^2} + \cdots,$$

where the higher terms are usually small enough to be safely ignored. If we let
$m = M' - M$ be the increase in mass from (3), we have

$$m = \frac{Mv^2}{2c^2} + \cdots$$

We also know (from high school physics) that the kinetic energy of our moving
object is

$$E = \frac{Mv^2}{2}$$

(Relativity says that this formula is not quite right. It should have higher terms
similar to the ones in (4), which are negligibly small in ordinary experience.)

Combining (4) and (5) gives the famous equation

$$E = mc^2.$$