Goals:

(1) To study the convergence and divergence of series.
(2) To study the size of the remainder term, which arises when estimating the sum of a series by its partial sum.
(3) Learn how to use integrals to estimate sums and vice-versa.

Problem 1. Suppose that the series $\sum a_n$ has positive terms and its partial sums $s_N$ satisfy $s_N < 1000$ for all $N$. Explain why $\sum_{n=1}^{\infty} a_n$ must be convergent.

Hint: What are the ways in which a sequence can diverge?

Problem 2.

(a) Compute this integral.
$$\int_1^{\infty} \frac{\ln x}{x^2} \, dx$$

(b) Show that the series
$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

is convergent. How do problem 2 and part (a) apply to this example?

(c) Find an upper bound for the error in the approximation $s \approx s_N$.

(d) What is the smallest value of $N$ such that this upper bound for the error is less than 0.05?

(e) Find $s_N$ for this value of $N$. 
(f) To restate the obvious, if you were to approximate this sum on a computer or a calculator to within plus or minus 0.05 how many terms would you have to add up?

Problem 3.

(a) Show that if $s_n$ is the $n$th partial sum of the harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$, then

$$s_n \leq 1 + \ln n.$$ 

(b) The harmonic series diverges, but very slowly. Use part (a) to find an upper bound for the sum of the first million terms, and also the sum of the first billion terms.

Problem 4.

For what value of $p$ does the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converge?