1. Power series and Taylor series

A power series provides an important new way to define a function. (Polynomials and quotients of polynomials are the most basic ways to define functions.)

It turns out that most known functions can be defined using a power series. The Taylor series formula tells how to discover the power series which defines a function by calculating the higher derivatives of the function. If just the first few terms of this series are used, then this Taylor polynomial can be used to calculate the values of a function to a certain accuracy. The error can be estimated to insure that the accuracy is sufficient (e.g. at least 8 places of accuracy for most calculators).

While the basic facts about power series, there are some subtleties that take a while to absorb. This workshop is designed to help.

Write a short paragraph discussing the concepts in the first three questions:

1. Domains and radius of convergence. In what ways are \( f(x) = 1/(1 - x) \) and \( g(x) = 1 + x + x^2 + x^3 \ldots \) the same function? Consider \( f(1/2) \) and \( g(1/2) \) and also \( f(2) \) and \( g(2) \). (A function is a rule or procedure for producing a unique output from an input together with a set of allowable inputs (the domain). Two functions are equal if their domains are equal and the same inputs produce the same outputs.)

2. How do you determine radius of convergence? Why do power series converge only on an interval? Why don’t some series converge for all \( x \) between \(-1\) and \(1\) and also for \( x \) between \(2\) and \(3\)?

3. Why is it called radius of convergence?
4. If you use the first 5 terms (up to $x^4$) to estimate $e^{-0.5}$ how accurate is the result. (There are two estimates of the accuracy that you can use in this case.) How accurately does this polynomial represent $e^x$ on the range of $x$ between $-0.5$ and $0.5$? What’s the worse case?

5. If you use the first five terms on what range of $x$ is the accuracy better than $10^{-5}$?

6. How many terms do you need to use to ensure that the accuracy is better than $10^{-5}$ for all $x$ between $-0.5$ and $0.5$?

7. Calculate the taylor series of $f(x) = x^2 e^x$. What’s the fastest technique for doing this? What is the easiest way to determine the 5th derivative of $f$?

8. Determine the first 8 terms of the Taylor series for $g(x) = \frac{1}{(1+x)^2}$. What is the eighth derivative of $xg(x)$?

9. How do you make sense of the Euler identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$? (see p. A56)

10. The Euler identity shows how the law of exponents and the angle sum formulas are related. Calculate an identity for $\cos(2a+b)$ and for $\sin(2a+b)$.

11. If a water wave with length $L$ moves with velocity $v$ across a body of water with depth $d$, as in the figure, then

$$v^2 = \frac{gL}{2\pi} \tanh \frac{2\pi d}{L}$$

(a) If the water is deep, show that $v \approx \sqrt{gL/(2\pi)}$.

(b) If the water is shallow, use the Maclaurin series for tanh to show that $v \approx \sqrt{gd}$. Thus, in shallow water the velocity of a wave tends to be independent of the length of the wave.

(c) Use the alternating series estimation theorem to show that if $L > 10d$ then the estimate $v^2 \approx gd$ is accurate to within $\pm 0.014gL$. 