Goals:

1. Observe that word problems can lead to integrals requiring trig substitutions.

2. Practice finding anti-derivatives using trig substitution techniques and learning to recognize patterns in the solution methods.

**Problem 1.** Suppose that a tank of water has a cylindrical shape with height 7 meters, and the radius of the top is 2 meters. Now suppose that someone moves the tank so that it is resting on its side. The tank is completely full of water.

1. Compute the work needed to pump the water up to a level 2 meters higher than the top of the tank.

2. What is the total mass $M$ of the water?

3. How much work would be done in raising an object of mass $M$ from the center of the tank to a level 2 meters higher than the top of the tank?

4. Compare the integral of work obtained by considering the work required to lift each horizontal slab with the center of mass approach.

5. Instead of using the height variable $y$ to specify the height of the slab, whose work you are calculating, use the angle $\theta$ as show in the diagram on the next page. What does the integral look like now? Is it easier to compute? Compare this with “trig substitution” techniques.
Problem 2. Evaluating these integrals involves first simplifying the integrand, using trig identities or the substitutions of the form $x = \cos \theta$, $x = \tan \theta$, etc. Try to detect patterns that you can use to speed up the process for simplifying these integrals. – as an example $\sin^n \theta \cos^m \theta$ is easier to simplify if one of $n$ or $m$ is odd. Why? What are your options if they are both even?

a. 
\[ \int \sin^2(x) \cos^5(x) \, dx \]

b. 
\[ \int \frac{\sec^6 x}{\tan^2 x} \, dx \]

c. 
\[ \int x \sec(x) \tan(x) \, dx \]

d. 
\[ \int \frac{dx}{\sqrt{x^2 - 4x}} \]

e. 
\[ \int \frac{x^2}{(4 - x^2)^{3/2}} \, dx \]

Problem 3. Use integration by parts and trig identities to show that
\[ \int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{(n-1)}{n} \int \sin^{n-2} x \, dx \]

Does this help solve $\int \sin^2 x \cos^2 x \, dx$? What are alternative methods for solving this?