HISPANOLA

150 miles

10 points
250 miles
20 points
320 miles
40 points
400 miles

MATH 162 10-9-09
How to measure the length of a curve

\[ y = f(x) \quad a \leq x \leq b \]

Total length of line segments approximates the length of the curve.
\[ \Delta s = \text{length of line} \]

\[ \Delta s = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \Delta x \]

Want to take limit as \( \Delta x \to 0 \)

We get

\[ s = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

length of curve from \((a, f(a))\) to \((b, f(b))\)
$S = \int_a^b \sqrt{1 + (y')^2} \, dx$ 

This integral can be \textit{NASTY} 

e.g. $y = x^3$, $y' = 3x^2$  

$S = \int_a^b \sqrt{1 + 9x^4} \, dx$  

This integral cannot be expressed in terms of \textit{familiar functions}.

Similar thing happens with an ellipse
Find the arc length of curve of radius $y = \sqrt{m^2 - x^2}$, $0 \leq x \leq m$

We know the length is

$$s = \int_0^m \frac{m \, dx}{\sqrt{m^2 - x^2}}$$

$$y' = \frac{1}{\sqrt{m^2 - x^2}}$$

$$1 + (y')^2 = 1 + \frac{x^2}{m^2 - x^2}$$

$$s = \int_0^m \frac{m \, dx}{\sqrt{m^2 - x^2}}$$
\[ S = \int_0^m \frac{mdx}{\sqrt{m^2 - x^2}} \]
\[ = \pi \int_0^{\pi/2} \frac{m \cos \theta \, d\theta}{\sqrt{m^2 - m^2 \sin^2 \theta}} \]
\[ = m \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2 \theta}} \]
\[ = m \cdot \frac{\pi}{2} \text{ CORRECT} \]
Find the arc length of the parabola $y = x^2$ between $(0,0)$ and $(2,4)$.

- Blue line has length $2\sqrt{5} \approx 4.472$.
- Red lines have lengths $\sqrt{2} + \sqrt{10} \approx 4.576$. 

$(0,0)$ $(2,4)$
\[ y = x^2 \quad y' = 2x \quad 1 + y'^2 = 1 + 4x^2 \]

\[ S = \int_0^2 \sqrt{1 + 4x^2} \, dx \]

\[ = \int_0^\frac{\arctan 4}{2} \sec^3 \theta \, d\theta \]

\[ x = \frac{\tan \theta}{2} \]

\[ dx = \frac{\sec^2 \theta \, d\theta}{2} \]

\[ \theta = \arctan 2x \]

\[ \sqrt{1 + 4x^2} = \sec \theta \]
\[ I = \int \sec^3 \theta \, d\theta \]

\[ = \sec \theta \tan \theta - \int \sec \theta \sec^2 \theta \, d\theta \]

\[ = \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) \, d\theta \]

\[ = \sec \theta \tan \theta - \int \sec^3 \theta \, d\theta + \int \sec \theta \, d\theta \]

\[ = \sec \theta \tan \theta - I + \ln |\sec \theta + \tan \theta| + C \]
\[ 2I = \sec \theta \tan \theta + \ln | \sec \theta + \tan \theta | + C \]
\[ I = \frac{\sec \theta \tan \theta + \ln | \sec \theta + \tan \theta |}{2} + C \]
\[ S = \frac{1}{2} \int_{0}^{\frac{\pi}{4}} \sec \theta \tan \theta \, d\theta \]
\[ = \frac{\sec \theta \tan \theta + \ln | \sec \theta + \tan \theta |}{4} \bigg|_{0}^{\frac{\pi}{4}} \]
\[ = \frac{4}{\sqrt{17}} + \ln \left( \frac{4 + \sqrt{17}}{4} \right) - 0 \]