Cheat sheet allowed

Exam at 8am

Topics covered

Chapter 10

Surface area

Parametric equations

Tangent line

Second derivative

Concavity
Polar co-ordinates

Area

Arc length

Sequences

Surface area $y = f(x)$

$$ds = \sqrt{1 + f'(x)^2} \, dx$$

rotate about x-axis
Area of slice is $2\pi y \Delta x$

$$S^1 = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} \, dx$$

E.g., $WW 7.4$  \quad $y = 4x^3$  \quad $y' = 12x^2$

$$S = \int_a^b 2\pi y \sqrt{1 + (12x^2)^2} \, dx$$

$$= 2\pi \left[ \left(4x^3 \sqrt{1+144x^4}\right) \right]_a^b$$

$m = 1+144x^4$

$dm = 576x^3 \, dx$

$x^3 \, dx = \frac{dm}{576}$
Parametric equations

\[ x = x(t), \quad y = y(t) \]

How to find the tangent line at time \( t \):

Slope of tangent line

\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = m \]

slope of tangent line

\( (x_0, y_0) \)

\( (x(t), y(t)) \)
Equation for line of slope in them $(x_0, y_0)$ is:
\[ \frac{y - y_0}{x - x_0} = m \]

WIV 8.1
\[ x = \cos^2 t \quad y = 6 \sin^2 t \]

When is the tangent line vertical?
\[ \frac{dx}{dt} = -9 \sin t \cos t \quad \frac{dy}{dt} = 16 \sin t \cos t \]
\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{16 \sin t \cos t}{-9 \sin t \cos t} = -\frac{16}{9} \]

When is this 0? NEVER Tangent is never horizontal.
When is $m$ undefined? When $t = \pi/2$ or $3\pi/2$.

In these cases $\cos t = 0$, so $x = 0$ is undefined and the tangent line is vertical.

When is curve concave upward/downward? i.e. when is $\frac{d^2y}{dx^2}$ positive/negative?

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dx} \right) / \frac{dt}{dx}$$
In our example \( \frac{dy}{dx} = 9 \sin t \cos^5 t \)

\( \frac{dy}{dx} = \frac{-16}{9} \sec^7 t \)

\( \frac{d}{dt} \left( \frac{dy}{dx} \right) = -\frac{7 \times 16}{9} \sec^6 t \sec t \tan t \tan t \)

\( = -\frac{112}{9} \sec^2 t \tan t \tan t \)

\( \frac{d^2 y}{dx^2} = \frac{d(\frac{dy}{dx})/dt}{dx/dt} = -\frac{112 \sec^2 t \tan t \tan t}{9 - 9 \sin t \cos^6 t} \)

\( = -\frac{112}{81} \sin t \cos^6 t \)
\[
\frac{112}{81} \frac{1}{\cos^6 t} = \frac{112}{81} \sec^6 t
\]

This is positive whenever it is defined, i.e., when \( t \neq \frac{\pi}{2} \) or \( 3\pi/2 \).

Curve is concave upward.

Polar co-ordinates:

\[
x = r \cos \theta
\]
\[
y = r \sin \theta
\]
For \( n = f(B) \),

Area of shaded region is

\[
\int_a^b \frac{m^2}{2} \, d\theta
\]

Length of blue section of curve

\[
L = \int_a^b \sqrt{m^2 + f'(B)^2} \, d\theta
\]
Arc length for parametric equations

\[ x = x(t) \quad y = y(t) \]

Length of curve is

\[ L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

E.g. Circle of radius \( r \)

\[ x = r \cos t \quad y = r \sin t \]

\[ 0 \leq t \leq 2\pi \]

\( r = \text{constant} \)
\[ \frac{dx}{dt} = -n \sin t \]
\[ \frac{dy}{dt} = n \cos t \]

\[ (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = n^2 \sin^2 t + n^2 \cos^2 t = n^2 \]

\[ \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = n \]

\[ s = \int_{0}^{2\pi/n} n \, dt = n \pi \left[ t \right]_{0}^{2\pi/n} = 2\pi/n \]
Sequences

Given a sequence \( a_1, a_2, a_3, \ldots \)

we want to find its limit

\[ L = \lim_{n \to \infty} a_n. \]

Tools:

- Sandwich method

Suppose we have 3 sequences \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) with

\[ a_n \leq b_n \leq c_n \]

for each \( n \).
Then \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \leq \lim_{n \to \infty} c_n \).

E.g., \( b_n = \frac{\sin n}{2^n} \). We know \(-1 \leq \sin n \leq 1\).

So \( -\frac{1}{2^n} \leq b_n \leq \frac{1}{2^n} \).

\( \lim_{n \to \infty} \frac{1}{2^n} = 0 \) = \( \lim_{n \to \infty} -\frac{1}{2^n} \). This means

\( 0 \leq \lim_{n \to \infty} \frac{\sin n}{2^n} \leq 0 \) \( \Rightarrow \lim_{n \to \infty} b_n = 0 \).
L'Hopital's rule

Suppose we want

\[ L = \lim_{x \to \infty} \frac{f(x)}{g(x)} \]

where \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty \)

L'Hopital's rule

\[ L = \lim_{x \to k} \frac{f'(x)}{g'(x)} \]

Same rule holds if \( x \to k \) and if

\[ \lim_{x \to k} f(x) = \lim_{x \to k} g(x) = 0 \]
Suppose \( a_n = \frac{f(n)}{g(n)} \) where \( f, g \) are as above.

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}
\]

e.g. \( a_n = n^2 e^{-n} = \frac{n^2}{e^n} \)

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{e^n} = \lim_{n \to \infty} \frac{2n}{e^n} = \lim_{n \to \infty} \frac{2}{e^n} = 0
\]