Talks on Riemann Hypothesis
1:00  CSB 209
2:30  Overugen 108  Soms

Ratio test for a series \( \sum a_n \)

Let \( R = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \)

If \( R < 1 \) and \( |R| < 1 \), the series converges absolutely.
$\sum |a_n|$ converges.

If $|R| < 1$, series diverges.

If $|R| = 1$, test is inconclusive.

Example

$$\sum_{n=1}^{\infty} \frac{n^4}{n!}$$

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

$$a_n > \frac{n^4}{n!} \left(\frac{(n+1)!}{n!}\right) = n+1$$

$$5! = 120$$

$$10! = 3.6288 \times 10^6$$

$$20! = 2.43 \times 10^{18}$$

$$50! = 3.04 \times 10^{64}$$
\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)^4 / (n+1)!}{n^4 / n!} \quad = \frac{(n+1)^4}{n^4} \frac{1}{(n+1)!} \\
= \left(1 + \frac{1}{n}\right)^4 \frac{1}{(n+1)!} \\
R = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^4 \frac{1}{(n+1)} = 0
\]
Root Test: Let \( L = \lim_{n \to \infty} |a_n| \)

This limit may or may not exist.

If it does, then

i) If \( L < 1 \), series converges absolutely.

ii) If \( L > 1 \), series diverges.

iii) If \( L = 1 \), test is inconclusive.
Example

Geometric series

\[ \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2 \]

\[ a_n = \frac{1}{2^n} \quad \sqrt[n]{\left| a_n \right|} = \sqrt[n]{1/2^n} = \frac{1}{2} \]

\[ L = \lim_{n \to \infty} \sqrt[n]{\left| a_n \right|} = \frac{1}{2} \]

Root test says this series converges.

Similarly, for any geometric series.
Example \[ \sum_{n=1}^{\infty} \left( \frac{2n+3}{3n+2} \right)^n = \frac{5}{3} + \left( \frac{7}{8} \right)^2 + \left( \frac{9}{11} \right)^3 \ldots \]

Ratio test fails here

\[ \lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \left( \frac{2n+3}{3n+2} \right)^n = \frac{2n+3}{3n+2} \]

\[ L = \lim_{n \to \infty} \left( \frac{2n+3}{3n+2} \right) = \frac{2}{3} \]

Hence the series converges.
Power series

\[ \sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \ldots \]

For a given value of \( x \), this series may or may not converge. Apply the ratio test.

\[ \frac{a_{n+1}}{a_n} = \frac{x^{n+1}/2^{n+1}}{x^n/2^n} = \frac{x}{2} \]

\[ R = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{x}{2} \]
By the ratio test, this converges if 
\[ \left| \frac{2^{n+1}}{2^n} \right| < 1 , \text{ i.e. if } |x| < 2 . \]
It diverges if \( |x| > 2 \).
Test is inconclusive if \( |x| = 2 \).
When it converges, it is a geometric series 
\[ 1 + x + x^2 + x^3 + \ldots \]  \text{ where } x = \frac{x}{2} .
The sum is \[ \frac{1}{1-x} = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x} . \]
WHOH CARES!

Calculator says
\[ \sin (15.37^\circ) = 0.33156263 \]

How does it know this?

We will see later that
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]
\[ = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)!} x^{2n+1} \]
Things we will learn about this series

1. It converges for all \( x \) (ratio test)

2. If we want to insure the error term is \( < \) (your choice), we can figure how terms we need in a partial sum.