Power series

\[ \sum a_n x^n \text{ or } \sum a_n (x-a)^n \]

Convergence depends on the value of \( x \). Use the ratio test.

Examples:

\[ \sum 2^n x^n = 1 + 2x + 4x^2 + 8x^3 + \ldots \]

\( n \text{th term} = 2^n x^n \)

\[ \sum^n 2^n x^n \]

\( n = 0 \)
\[ R = \lim_{n \to \infty} \left( \frac{(n+1)\text{th term}}{n\text{th term}} \right) = \lim_{n \to \infty} \left( \frac{2^{n+1}x^{n+1}}{2^n x^n} \right) \]

\[ = \lim_{n \to \infty} 2x = 2x \]

Series converges if \( |2x| < 1 \)

\[ |x| < \frac{1}{2} \quad \text{i.e.} \quad \frac{1}{2} < x < \frac{1}{2} \]

Diverges if \( |x| \geq \frac{1}{2} \)

\( \frac{1}{2} \) here is called the radius of convergence.
Example \[ \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \ldots \]

\[ \frac{n\text{ th term}}{n-1\text{ th term}} = \frac{n!x^n}{(n-1)!x^{n-1}} = \frac{n+1}{n} \cdot x \]

\[ R = \lim_{n \to \infty} \frac{(n+1)x}{n} = \lim_{n \to \infty} \begin{cases} 0 & \text{if } x = 0 \\ \infty \end{cases} \]

Series converges if \( x = 0 \) diverges if \( x \neq 0 \)

Radius of convergence is 0
Example \[
\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots.
\]

\[R = \lim_{n \to \infty} \left( \frac{x^{n+1}/(n+1)!}{x^n/n!} \right) = \lim_{n \to \infty} \left( \frac{x}{n+1} \right) = 0\] for any \(x\).

This series converges for all \(x\). Radii of convergence is \(\infty\).
Example: \[ \sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n} = 1 + \frac{x-3}{2} + \frac{(x-3)^2}{4} + \ldots \]

\[
R = \lim_{n \to \infty} \left( \frac{(x-3)^{n+1}}{2^{n+1}} \right) = \lim_{n \to \infty} \left( \frac{x-3}{2} \right) = \frac{x-3}{2}
\]

Series converges if \( \left| \frac{x-3}{2} \right| < 1 \)

\[
-2 < x-3 < 2 \quad \Rightarrow \quad 1 < x < 5
\]

Series diverges if \( x < 1 \) or \( x > 5 \)
When a power series converges, it converges to a function of \( x \). Which function of \( x \)?

Recall the geometric series.

\[
\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots
\]

converges to \( \frac{1}{1-x} \) if \( |x| < 1 \).

**Example** Find a power series that converges to \( \frac{1}{2 + x} \) for “small” \( x \).
\[
\frac{1}{2 + x} = \frac{1}{2 \left(1 + \frac{x}{2}\right)} = \frac{1}{2} \left( \frac{1}{1 - \left(-\frac{x}{2}\right)} \right) = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}
\]

\[
= \frac{1}{2} \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots\right)
\]

\[
= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \ldots
\]

This series converges when \( \left|\frac{-x}{2}\right| < 1 \)

\(-1 < \frac{-x}{2} < 1 \)

\(-2 < -x < 2 \)

\(-2 > x > -2 \)
Conclusion: For $-2 < x < 2$, the series $\sum_{n=0}^{\infty} \frac{(-x)^n}{2^{n+1}}$ converges to $\frac{1}{2+x}$.

The radius of convergence is 2.

$$\sum_{n=0}^{\infty} n^n = \frac{1}{1-n} \text{ for } |n| < 1$$

Geometric series
Example: Find a power series converging to \( \frac{1}{1+x^2} \) for "small" \( x \).

Let \( a_n = (-x^2)^n \) and we get
\[
\frac{1}{1+x^2} = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-x^2)^n
\]

\[= 1 - x^2 + x^4 - x^6 + \ldots \]

for \( |x^2| < 1 \), i.e. if \( |x^2| < 1 \), \(-1 < x < 1 \).
If a power series converges to \( f(x) \), then we can differentiate/integrate term by term and get a series converging to \( f'(x) \) or \( \int f(x) \, dx \).

\[
\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{for} \quad -1 < x < 1
\]

We want to integrate both sides. On the right we have
\[1 - x^2 + x^4 - x^6 + x^8 - \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n}\]

Integrating termwise gives
\[\frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \frac{x^7}{9} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\]

On the left we get \text{arctan } x.

Ratio test shows this series converges for \(|x| < 1\), diverges for \(|x| > 1\).
\[
\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{for} \quad |x| < 1
\]

What happens for \( x = 1 \)?

\[
\frac{1}{1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots
\]

converges by all series test