Taylor's inequality

Taylor series for \( f(x) \) at \( a \)
\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

Taylor polynomial
\[
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k
\]

You can compute this...
$$R_n(x) := f(x) - T_n(x)$$

- with Taylor remainder

Taylor's inequality. Assume \( |x-a| \leq d \)

\[ \text{You choose } d. \]

Find a \( \neq M \) s.t. \( |f^{(n+1)}(x)| \leq M \)

whenever \( |x-a| \leq d \).

Then \( |R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!} \)
Example \( f(x) = x^{1/2} = \sqrt{x} \quad a = 4 \)

\[
T_3(x) = 2 \left(\frac{x-4}{4}\right) - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} + \ldots
\]

Assume \( 4 \leq x \leq 5 \)

Warmup \( x = 5 \quad \sqrt{5} = 2.236 \)

\[
T_3(x) = 2 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} + \ldots
\]

\[
= 2 + 0.25000 + 0.015625 + 0.001953 + \ldots
\]

\[
= 2.235703
\]
\[ f(x) = x^{1/2} \quad 4 \leq x \leq 5 \]
\[ f'(x) = \frac{1}{2} x^{-1/2} \]
\[ f''(x) = -\frac{1}{4} x^{-3/2} \]
\[ f'''(x) = \frac{3}{8} x^{-5/2} \]
\[ f^{(4)}(x) = \frac{15}{16} x^{-7/2} \]
\[ f^{(4)}(4) = -\frac{15}{16} \cdot 4^{-1/2} = -\frac{15}{16} \cdot \frac{1}{2^{5/2}} = -\frac{15}{16} \cdot \frac{1}{2^{5/2}} = -\frac{15}{16} \cdot \frac{1}{2^2} = -\frac{15}{16} \cdot \frac{1}{4} = -\frac{15}{64} = -0.234375 \approx -0.2344 \]
\[ M = 0.075 \]
Taylor's inequality says
\[ |R_k(x)| \leq \frac{M}{k!} \]
where \( k = 3 \)

Improper integral
2 types of improperity
① \( a = \infty \) or \( b = -\infty \)
② \( f(x) \) might go to \( \infty \) somewhere

\[ \int_a^b f(x) \, dx \]
In both cases, find a way to write the integral as a limit of proper integrals.

E.g., \[ \int_{\frac{1}{x^2}} \frac{1}{x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x} \, dx \]

\[ = \lim_{b \to \infty} \left[ \frac{-1}{x} \right]_{1}^{b} = \lim_{b \to \infty} \left(1 - \frac{1}{b}\right) = 1 \]

This integral converges.
Second kind of improperity

\[ \int_{0}^{1} \frac{dx}{x^2} = \lim_{a \to 0^+} \int_{a}^{1} \frac{dx}{x^2} \]

\[ = \lim_{a \to 0^+} \left[ -\frac{1}{x} \right]_{a}^{1} = -\lim_{a \to 0^+} \left( \frac{1}{a} - 1 \right) \]

\[ = \infty \]

Integral diverges
\[
\lim_{n \to \infty} \frac{x^n}{n!}
\]

Let \( C_n = \frac{x^n}{n!} \) so

\[
\frac{C_{n+1}}{C_n} = \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1}
\]

If \( n \) is large enough, this is very small
Suppose \( x = 328 \)

\[ C_n = \frac{328^n}{n!} \]

Consider \( n = 400 \)

\[ 400! = 6.4 \times 10^{668} \]

For \( n > 400 \),

\[ \frac{C_{n+1}}{C_n} = \frac{328}{n} \leq \frac{328}{400} = 0.802 \]

\[ C_{400} \text{ is huge} \]

\[ C_{400+k} < C_{400} (k < 802) \]
By making \( k \) very large we can make this \( \# \) as small as we want.

**WORK Problem**

Pool diameter = 4', sides = 5', depth of \( H_2O = 4' \), density = 62.5 lbs/ft\(^3\).
Let $x$ be the distance from the top \[1 \leq x \leq 5\]

How much does this layer of $H_2O$ weigh?

Weight = density $\times$ volume

$= 1 \text{ lb/ft}^3 \times \text{area} \times \text{thickness}$

$= \frac{62.5 \text{ lb}}{\text{ft}^3} \times 144 \pi \frac{\text{ft}^2}{\text{ft}^3} \times \Delta x \text{ ft}$

Work required to lift this water
= \text{wt \times distance}
= 62.5 \times 144 \frac{x}{K}

\text{Total work} = \int_{1}^{5} K x \, dx

= \left. K x^2 \right|_{1}^{5} / 2
= \frac{K (5^2 - 1^2)}{2}
= K \frac{24}{2} = 12 K
This answer is in ft-lbs.
In metric units we need to multiply by \( \frac{9.8 \text{ m}}{\text{sec}^2} = g \).
We get an answer in term of Joules
\[ 1 \text{J} = \frac{m \cdot \text{kg}}{\text{sec}^2} \]
Integral test for series
\[ \sum_{n=0}^{\infty} a_n \quad \text{with} \quad a_n = f(n) \geq 0 \]

If \( f(n) \) is decreasing

Series converges \( \Rightarrow \)

\( \int_{a}^{\infty} f(x) \, dx \) converges

e.g. \( f(x) = \frac{\ln x}{x} \)
\[
\int_{1}^{\infty} \frac{\ln x}{x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{\ln x}{x} \, dx
\]

\[
= \lim_{b \to \infty} \left[ \ln x \cdot \ln x \right]_{1}^{b} = \lim_{b \to \infty} \left[ \ln x^2 \right]_{1}^{b} = \lim_{b \to \infty} (\ln b^2 - \ln 1) = \lim_{b \to \infty} 2 \ln b = \infty
\]
Series and integral converge.

Power series

\[ \sum_{n=0} C_n x^n \]

To find radius of convergence, use the ratio test.

The ratio test for the \( n \)th term:

\[ \frac{a_{n+1}}{a_n} = \frac{C_{n+1} x^{n+1}}{C_n x^n} = x \lim_{n \to \infty} \frac{C_{n+1}}{C_n} \]
Let $L_0 = \lim_{n \to \infty} \frac{C_{n+1}}{C_n}$

$L = xL_0$, series converges if $|L| < 1$
series diverges if $|L| > 1$

Test is inconclusive if $|L| = 1$
series converges if $|x| < \frac{1}{L_0}$

etc.

When $L = \pm 1$, try the integral test
or alternating series test.
\[ \sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{n+1} \]

\[ C_n = \frac{(-1)^n}{n+1} \]

\[ \frac{C_{n+1}}{C_n} = \frac{(-1)^{n+1}/(n+1)}{(-1)^n/n} = -\frac{n}{n+1} \]

\[ L_0 = \lim_{n \to \infty} \frac{C_{n+1}}{C_n} = -1 \]

\[ L = (x+1) L_0 = - (x+1) \]

Series converges if
\[ |x+1| < 1 \]
-1 < x+1 < 1
-2 < x < 0
if \( x > 0 \) or \( x < -2 \)

\[
\sum \frac{(-1)^n (x+1)^n}{n+1}
\]

Ratio test does not tell us what happens when \( x = 0 \) or \( x = -2 \),

For \( x = 0 \), this is \( \sum \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \).
This converges by the alternating series test.

For \( n = -2 \), \( x + 1 = -1 \) and the series is

\[
\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots
\]

Diverges by integral test.

Taylor series for \( f(x) \) at \( a \)

\[
f(x) = \sum_{n \geq 0} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]
For \( x = \sin x \), as \( x \to 0 \), one has \( f^{(n)}(0) \) is easy to calculate. It is \( O(x^n) \) depending on \( n \).

How to find series for arctan \( \frac{\pi}{4} \)?

\[
f'(x) = \frac{1}{1 + x^2}
\]

Geometric series formula says

\[
\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \ldots
\]

\[
= \sum_{m=0}^{\infty} (-1)^m x^{2m}
\]

Converges for \( |x| < 1 \).
To get arctan \( x \), integrate this series term by term. We get
\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]
Ratio test says it converges for \(|x| < 1\) and diverges for \(|x| \geq 1\).
If \( x = \pm 1 \), series is
\[
\pm (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots)
\]
Converges by alternating series test.
\[ \int \sqrt{(6-x^2)(11-x)} \, dx \]

\[ = \int 4 \cos^2 \theta (11-4 \cos^2 \theta) \, d\theta \]

\[ = \int 176 \cos^2 \theta \, d\theta \]

\[ = 56 \int 2 \cos^2 \theta \, d\theta \]

\[ = 56 \int (1 + \cos 2\theta) \, d\theta \]

\[ = 28 \theta + 14 \sin 2\theta + C \]
Area between \( y = x^2 \) and \( y = 2x \) revolved around the line \( x = 2 \)

Volume of the shell

Let x thickness

\( x \cdot 2\pi \) radius

\[ = (2x - x^2) \cdot 2\pi \cdot (2-x) \Delta x \]

\[ = 2\pi x (4 - 4x + x^2) \Delta x \]

\[ V = \pi \int_0^2 \left( 2\pi x (4 - 4x + x^2) \right) dx \]
Surface area for line: \( y = 2x \) \( \frac{dy}{dx} = 2 \)

\[ \Delta s = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \Delta x \]

\[ = \sqrt{5} \Delta x \]

Surface area of slice

\[ = 2\pi \text{ radius } \Delta s \]

\[ = 2\pi (2-x) \sqrt{5} \Delta x \]

\[ S = \int_0^2 2\pi \sqrt{5} (2-x) \, dx \]
Surface area of paraboloid

\[ \Delta S = \sqrt{1 + (2y')^2} \, \Delta x \]

\[ = \sqrt{1 + 4x^2} \, \Delta x \]

\[ S' = \int_{0}^{\sqrt{2}} (2\pi (2-x) \sqrt{1 + 4x^2}) \, dx \]
Taylor series for $e^{2x}$ at $a = 1$

$f(x) = e^{2x}$

$f^{(n)}(x) = 2^n e^{2x}$

$f^{(n)}(1) = 2^n e^2$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

$$= \sum_{n=0}^{\infty} 2^n e^2 \frac{(x-1)^n}{n!}$$
\[ e^x = (1 + 2(x-1) + \frac{4}{2} (x-1)^2 + \frac{8}{6} (x-1)^3 + \ldots) \]
Parametric equation

\[ x = 2(t - \sin t) \quad 0 \leq t \leq 2\pi \]

\[ y = 2 \left(1 - \cos t\right) \]

When is tangent line vertical or horizontal?

\[ \frac{dx}{dt} = 2(1 - \cos t) \]

\[ \frac{dy}{dt} = 2\sin t \]

0 when \( t = 0, 2\pi \)
M. of tangent line is

\[ y' = \frac{dy}{dt} = \frac{2 \sin t}{t(1 - \cos t)} \]

\[ = \begin{cases} 0 & \text{when } t = \pi \\ \text{horizontal tangent} & \text{for } t = 0 \\ 2 & \text{for } t \to 0 \end{cases} \]

At \( t = 0 \)

\[ y' = \lim_{t \to 0} \frac{\sin t}{1 - \cos t} = \lim_{t \to 0} \frac{\cos t}{\sin t} \]

\[ = \infty \]
Tangent line is vertical for $t = 0$ and $2\pi$.

$$\lim_{n \to \infty} \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{4} \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$= \frac{2}{4} = \frac{1}{2}.$$
\[
\sum_{n=0}^{\infty} (x-a)^n
\]

\[
= \frac{1}{1-(x-a)}
\]

for \(|x-a| < 1\)
Taylor series for $x^{-5}$ at $a=5$

$$
\sum C_n (x-5)^n
$$

$$
c_n = \frac{f^{(n)}(5)}{n!}
$$

$$
c_0 = 5^{1/2}
$$

$$
c_1 = 5^{-1/2} / 2
$$

$$
c_2 = 5^{-3/2} / 4! = \frac{-5^{-3/2}}{8}
$$

$$
c_3 = 5^{-5/2} / 3! = \frac{-5^{-5/2}}{16}
$$

$$
f(x) = x^{1/2}
$$

$$
f'(x) = \frac{1}{2} x^{-1/2}
$$

$$
f''(x) = -\frac{1}{4} x^{-3/2}
$$

$$
f'''(x) = \frac{3}{8} x^{-5/2}
$$
\[ c_n = 5^{\frac{1}{2}} n \left( 1 \right)^n \ \text{is probably} \ \frac{5}{n!} \]

radius of convergence

\[ y = \cos x, \quad y = \sin 2x \quad 0 \leq x \leq \pi/2 \]

\[ \cos x = \sin 2x \]
\[ = 2 \sin x \cos x \]
\[ \sin x = 1/2 \quad x = \pi/6 \]
\[ y = 2x + 1 \]
\[ y = x^2 + 2x = x(x + 2) \]
\[ x^2 + 2x = 2x + 1 \]
\[ x^2 = 1 \]
\[ x = \pm 1 \]
\[ y = -1, 3 \]
\[ n = \frac{1}{2} + \cos \theta \]
\[ y = \frac{1}{2} + \cos \theta \]

\[ \cos \theta = -1/2 \]

\[ \theta = \frac{2}{3} \pi \]

\[ \text{Area of loop} \]

\[ 2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{M^2}{2} \, d\theta = \text{area of loop} \]