Solids of revolution

Problem 2.6: Rotate the region between 

\[ y = x^2 \quad \text{and} \quad y = 3x \]

about the line \( y = 9 \).
Rotating a vertical strip about $y = 9$ gives a washer or annulus.

inner radius $= 9 - 3x = r_1$

outer radius $= 9 - x^2 = r_2$
Area of annulus or washer: area of outer circle - area of inner circle

\[ = \pi \left( r_2^2 - r_1^2 \right) \]

\[ = \pi \left( (9-x^2)^2 - (9-3x)^2 \right) \]

\[ = \pi \left( (81-18x^2 + x^4) - (81-54x + 9x^2) \right) \]

\[ = \pi \left( x^4 - 27x^2 + 54x \right) \]

\[ V = \int_0^3 \text{area} \cdot dx = \pi \left( \int_0^3 (x^4 - 27x^2 + 54x) \right) dx \]

\[ = \pi \left. \left( \frac{x^5}{5} - 9x^3 + 27x^2 \right) \right|_0^3 \]
\[ = \pi \left( \frac{3^5}{5} - 9 \cdot 3^3 + 27 \cdot 3^2 \right) = \pi \left( \frac{243}{5} - 243 + 243 \right) \]

\[ = \frac{243}{5} \pi \]

2.10 A soda glass has the shape of \( y = \frac{5}{2} x^2 \) for \( 0 \leq x \leq 1 \), rotated about the y-axis (units are inches). You are emptying the soda at \( \frac{1}{2} \text{ in}^3/\text{sec} \). How fast is the level dropping when there
Is 3" of soda left in the glass?

rate of change = rate of change of volume of level

area = \( \pi x^2 \). What is \( x \) when \( y = 3? \)

\[
\begin{align*}
y &= 5x^2 = 3 \\
x &= \sqrt{\frac{3}{5}} \\
\text{area} &= 3 \pi \left( \frac{3}{5} \right) \\
\text{rate} &= \frac{1}{2} \text{ in}^3/\text{sec} \\
\text{area} &= \frac{5}{6 \pi} \text{ in}^2
\end{align*}
\]
Don't problem $x \geq b$

CIRCLE:

$x^2 + y^2 = a^2$

$y = \pm \sqrt{a^2 - x^2}$

$-a \leq x \leq a$

$0 < a < b$

Rotate the circle about the line
height of shell = \( 2 \sqrt{a^2 - x^2} \)

radius = \( b - x \)

area of shell = \( \text{height} \times \text{circumference} \)

= \( 2 \sqrt{a^2 - x^2} \times 2\pi (b - x) \)

\[ V = 4\pi \int_{-a}^{a} (b - x) \sqrt{a^2 - x^2} \, dx \]

\( \Rightarrow \) area of small circle (radius \( a \))

\( \times \) circumference of big circle (radius \( b \))

= \( \pi a^2 \times 2\pi b = 2\pi \sqrt{a^2 - b^2} \)