Math 162: Calculus IIA
First Midterm Exam ANSWERS
October 20, 2009

1. (20 points)
(a) (10 points) Find a partial fraction expansion for the function

\[
\frac{1}{x^3 - x^2 + 2x - 2}
\]

1. (b) (10 points) Calculate the integral

\[
\int \frac{dx}{x^3 - x^2 + 2x - 2}.
\]

Solution: (a) One notices that 1 is a root of the denominator. Polynomial division yields

\[x^3 - x^2 + 2x - 2 = (x - 1)(x^2 + 2),\]

so

\[
\frac{1}{x^3 - x^2 + 2x - 2} = \frac{1}{(x - 1)(x^2 + 2)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2}
\]

\[
= \frac{A(x^2 + 2) + (Bx + C)(x - 1)}{(x - 1)(x^2 + 2)} = \frac{Ax^2 + 2A + Bx^2 + Cx - Bx - C}{(x - 1)(x^2 + 2)}
\]

By comparing numerators we must have \(A + B = 0\), \(C - B = 0\) and \(2A - C = 1\). From this we get \(A = 1/3\), \(B = -1/3\) and \(C = -1/3\).

(b) To calculate this integral, use the partial fraction expansion from (a). One gets:

\[
\int \frac{dx}{x^3 - x^2 + 2x - 2} = \int \left( \frac{1}{3(x - 1)} - \frac{x + 1}{3(x^2 + 2)} \right) dx
\]

\[
= \frac{1}{3} \int \left( \frac{1}{x - 1} - \frac{x}{x^2 + 2} - \frac{1}{x^2 + 2} \right) dx
\]

\[
= \frac{1}{3} \left( \int \frac{dx}{x - 1} - \int \frac{x dx}{x^2 + 2} - \int \frac{dx}{x^2 + 2} \right).
\]
The first two integrals are done by substitution, $u = x + 1$ in the first integral and $u = x^2 + 2$ in the second integral. For the last summand we use the trigonometric substitution $x = \sqrt{2} \tan(\theta)$ and hence $dx = \sqrt{2} \sec^2(\theta) d\theta$. The last integral then becomes

$$\int \frac{dx}{x^2 + 2} = \int \frac{\sqrt{2} \sec^2(\theta) d\theta}{2 \sec^2(\theta)} = \int \frac{d\theta}{\sqrt{2}}.$$

substituting back $\theta = \arctan\left(\frac{x}{\sqrt{2}}\right)$ we get

$$\int \frac{dx}{x^2 + 2} = \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right).$$

Combining the three calculations we get

$$\int \frac{dx}{x^3 - x^2 + 2x - 2} = \frac{1}{3} \left( \ln|x + 1| + \frac{1}{2} \ln(x^2 + 2) + \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) \right) + C.$$

2. (20 points) Consider the curve $y = x^{3/2}$

(a) (10 points) Calculate the arc length function starting at $x = 0$.

2. (b) (10 points) Calculate the arc length from $x = 4$ to $x = 8$.

Solution: (a) $y' = \frac{3}{2} \sqrt{x}$, so by substituting $u = 1 + \frac{9}{4} x$ one gets

$$s(t) = \int_0^t \sqrt{1 + \frac{9}{4} x} \, dx = \frac{4}{9} \int_1^{1 + 9t/4} \sqrt{u} \, du = \frac{8}{27} u^{3/2} \bigg|_{1}^{1 + 9t/4} = \frac{8}{27} \left( 1 + \frac{9}{4} x \right)^{3/2} - \frac{8}{27}$$

for $t \geq 0$.

(b) By the definition of the arc length function, $s(4)$ is the arclength from $t = 0$ to $t = 4$
and \( s(8) \) is the arclength from \( t = 0 \) to \( t = 8 \), so the arc length from \( t = 4 \) to \( t = 8 \) is

\[
s(8) - s(4) = \frac{8}{27} \left( 19^{3/2} - 10^{3/2} \right) = \frac{8}{27} \left( 19\sqrt{19} - 10\sqrt{10} \right).
\]

3. **(20 points)** Consider region between the curve \( y = \sin^2 x \) for \( 0 \leq x \leq \pi \) and the \( x \)-axis.

(a) Find the volume of the solid of revolution about the \( x \)-axis.

3. (b) Find the volume of the solid of revolution about the \( y \)-axis.

**Solution:** (a) This is a washer method problem. We have

\[
V = \int_0^\pi \pi y^2 \, dx
\]

\[
= \pi \int_0^\pi \sin^4 x \, dx
\]

\[
= \pi \int_0^\pi \left( \frac{1 - \cos 2x}{2} \right)^2 \, dx
\]

\[
= \frac{\pi}{4} \int_0^\pi \left( 1 - 2 \cos 2x + \cos^2 2x \right) \, dx
\]

\[
= \frac{\pi}{4} \int_0^\pi \left( 1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) \, dx
\]

\[
= \frac{\pi}{8} \int_0^\pi \left( 3 - 4 \cos 2x + \cos 4x \right) \, dx
\]

\[
= \frac{\pi}{8} \left( 3x - 2 \sin 2x + \frac{\sin 4x}{4} \right) \bigg|_0^\pi
\]

\[
= \frac{3\pi^2}{8}.
\]
(b) This is a shell method problem. We have
\[
V = \int_{0}^{\pi} 2\pi xy \, dx \\
= 2\pi \int_{0}^{\pi} x \sin^2 x \, dx \\
= 2\pi \int_{0}^{\pi} x \left(1 - \cos 2x\right) \, dx \\
= \pi \int_{0}^{\pi} x \, dx - \pi \int_{0}^{\pi} x \cos 2x \, dx \\
= \frac{\pi^3}{2} - \pi \int_{0}^{\pi} x \cos 2x \, dx
\]
The remaining integral requires integration by parts with
\[
\begin{align*}
    u &= x & dv &= \cos 2x \, dx \\
    du &= dx & v &= \frac{\sin 2x}{2}
\end{align*}
\]
This gives
\[
\int_{0}^{\pi} x \cos 2x \, dx = \left. \int_{x=0}^{x=\pi} u \, dv \right|_{x=0}^{x=\pi} \\
= u v \bigg|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} v \, du \\
= \frac{x \sin 2x}{2} \bigg|_{x=0}^{x=\pi} - \int_{0}^{\pi} \frac{\sin 2x}{2} \, dx \\
= 0,
\]
so
\[
V = \frac{\pi^3}{2}.
\]

4. (20 points)

(a) (10 points) Use integration by parts to find a formula for
\[
\int x^n e^x \, dx \quad \text{in terms of} \quad \int x^{n-1} e^x \, dx
\]
(b) (10 points) Use this formula to find
\[ \int x^3 e^x \, dx. \]

**Solution:** (a) Let \( u = x^n \) and \( dv = e^x \, dx \), so \( du = nx^{n-1} \) and \( v = e^x \). Then we have
\[
\int x^n e^x \, dx = \int u \, dv = uv - \int v \, du = x^n e^x - n \int x^{n-1} e^x \, dx.
\]

(b)
\[
\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx
= x^3 e^x - 3 \left( x^2 e^x - 2 \int e^x \, dx \right)
= (x^3 - 3x^2)e^x + 6 \int e^x \, dx
= (x^3 - 3x^2)e^x + 6 \left( e^x - \int e^x \, dx \right)
= (x^3 - 3x^2 + 6x)e^x - 6 \int e^x \, dx
= (x^3 - 3x^2 + 6x - 6)e^x + C
\]

5. (20 points) Consider the integral
\[
\int \frac{dx}{\sqrt{4x^2 - 12x}}
\]

(a) (5 points) Write the quantity under the square root sign as a sum or difference of two squares.

(b) (5 points) Draw a right triangle in which one of the sides is the square root in the integer and another is a constant.

5. (c) (10 points) Evaluate
\[
\int_3^4 \frac{dx}{\sqrt{4x^2 - 12x}}.
\]

**Solution:** (a) \((2x - 3)^2 = 4x^2 - 12x + 9\) so \(4x^2 - 12x = (2x - 3)^2 - 3^2\).
(a) The triangle has hypotenuse $2x - 3$, adjacent side 3 and opposite side $\sqrt{4x^2 - 12x}$.

(c) We have

\[ \sqrt{4x^2 - 12x} = 3 \tan \theta \]
\[ 2x - 3 = 3 \sec \theta \]
\[ 2dx = 3 \sec \theta \tan \theta d\theta \]

so the indefinite integral is

\[
\int \frac{dx}{\sqrt{4x^2 - 12x}} = \frac{3}{2} \int \frac{\sec \theta \tan \theta}{3 \tan \theta} d\theta
\]
\[
= \frac{1}{2} \int \sec \theta d\theta
\]
\[
= \frac{1}{2} \log(\sec \theta + \tan \theta) + C
\]
\[
= \frac{1}{2} \log \left( \frac{2x - 3}{3} + \sqrt{4x^2 - 12x} \right) + C
\]

and

\[
\int_3^4 \frac{dx}{\sqrt{4x^2 - 12x}} = \frac{1}{2} \log \left( \frac{2x - 3}{3} + \sqrt{4x^2 - 12x} \right) \bigg|_3^4
\]
\[
= \frac{1}{2} \left( \log \left( \frac{5}{3} + \sqrt{52} \right) - \log \left( 1 + \sqrt{24} \right) \right)
\]
\[
= \frac{1}{2} \left( \log \left( \frac{5}{3} + 2\sqrt{13} \right) - \log \left( 1 + 2\sqrt{6} \right) \right)
\]