More about partial fractions.

This is a strategy for integrating rational functions, i.e.

\[
\frac{P(x)}{Q(x)} \quad \text{where } P(x) \text{ and } Q(x)
\]

are polynomials.

1. Need to factor \( Q(x) \).
2. Check to see if degree of \( P(x) \) is less than degree of \( Q(x) \). If not, divide...
$p(x)$ by $g(x)$ and get a quotient $q(x)$ and a remainder $m(x)$. Then

$$\frac{p(x)}{g(x)} = q(x) + \frac{m(x)}{g(x)}$$

where $\deg m(x) < \deg g(x)$.

E.g. \( \frac{20}{7} = 4 + \frac{2}{7} \)

What happens if $g(x)$ has repeated factors?

E.g. \( \frac{x^2 + 3}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} \)

The right-hand side is easy to integrate.
Brute force method

\[ R'(x) = \frac{A + B(x+1) + C(x+1)^2}{(x+1)^3} \]

\[ = \frac{A + (B(x+1) + C(x^2 + 2x + 1))}{(x+1)^3} \]

\[ = \frac{Cx^2 + (2C+B)x + (A+B+C)}{(x+1)^3} \]

This means

\[ C = 1 \]

\[ 2C + B = 0 \]

\[ A + B + C = 3 \]

\[ \begin{align*}
C &= 1 \\
2 + B &= 0 \\
A - 2 + 1 &= 3 \\
B &= -2 \\
A &= 4
\end{align*} \]
Heaviside method

\[
\frac{x^3 + 3}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}
\]

Multiply both sides by \((x+1)^3\)

\[
x^3 + 3 = A(x+1)^2 + B(x+1) + C
\]

Let \(x+1 = 0\), i.e. \(x = -1\)

\[
4 = A
\]

Subtract the \(A\) term from both sides of (1)

\[
\frac{x^3 + 3 - 4}{(x+1)^3} = \frac{13}{(x+1)^2} + \frac{C}{x+1}
\]

\[
= \frac{x^3 - 1}{(x+1)^3} = \frac{x^2 - 1}{(x+1)^3}
\]
\[
\frac{x+1}{(x+1)^3} = \frac{x-1}{(x+1)^2} + \frac{B}{(x+1)^2} + \frac{C}{x+1}.
\]

Find B here the same way we found A before.

Another possible wrinkle: \(f(x)\) could have an irreducible quadratic factor

\[
\frac{x+3}{x^4-1} = \frac{x+3}{(x^2-1)(x^2+1)} = \frac{x+3}{(x+1)(x-1)(x^2+1)}
\]
\[ \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \]

The number of constants is always the degree of the denominator.

Brute force method:

RHS of (3) = \[ \frac{A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1)}{(x+1)(x-1)(x^2+1)} \]

= expanded numerator

\[ \text{denom} \]

UGHH!

Hermannides method
\[ \frac{x+3}{x^4+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \]

To find \( A \), multiply by \( x+1 \) and set \( x+1 = 0 \), \( x = -1 \)

\[ \frac{x+3}{(x-1)(x^2+1)} = A + (x+1) \left( \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \right) \]

\( x = -1 \)

\[ \frac{-1+3}{(-2)(2)} = A = -1 \]

Use similar method to find \( B \)

\[ \frac{x+3}{(x+1)(x^2+1)} = B + \left( \frac{C}{x-1} + \frac{D}{x^2+1} \right) \]

\( x = 1 \)
Now solve \( \Box \) \hspace{1cm} \text{by} \hspace{1cm} \frac{c(x+1)}{x^2+1}

\[
\frac{c(x+1)}{x^2+1} = \frac{x+3}{x^2+1} - \frac{A}{x+1} - \frac{B}{x-1}
\]

\[
= \frac{x+3}{x^2+1} + \frac{1}{2(x+1)} - \frac{1}{x-1}
\]

\[
= \frac{2(x+3) + 1 \cdot (x-1)(x+1) - 2(x+1)x^2}{2(x+1)(x-1)(x^2+1)}
\]

\[
= \frac{2x + 6 + x^3 - x^2 + x - 1}{2x^2 + 2x - 2}
\]

\[
= \frac{2x+6 + x^3 - x^2 + x - 1 - 2x^3 - 2x^2 - 2x - 2}{\text{denom}}
\]
\[ \frac{-x^3 - 3x^2 + x + 3}{\text{denom}} = \frac{-(x+3)(x^2-1)}{2(x-1)(x+1)(x^2+1)} \Rightarrow \frac{-(x+3)}{2(x^2+1)} \]

\[
\frac{(x+1)}{x^2+1} = \frac{-(x+3)}{2(x^2+1)} \Rightarrow \frac{-(x+3)}{2}
\]

so \( C = -\frac{1}{2} \) and \( D = -\frac{3}{2} \)

The original unated problem:

\[ \int \frac{x+3}{x^4-1} \, dx \]
We know this is the same as

\[
A \int \frac{1}{x^2+1} \, dx + B \int \frac{1}{x^2+1} \, dx + C \int \frac{1}{x^2+1} \, dx
\]

**For 5**

\[
u = x^2 + 1
\]
\[
du = 2x \, dx
\]
\[
x = \frac{dy}{2}
\]

**For 7**

\[
\int \frac{dx}{x^2+1} = \arctan x + C
\]