MATH 162

2

\[ 1 + 2 + 3 + \ldots + 99 + 100 \]

\[ 100 + 99 + 98 + \ldots + 1 \]

\[ 101 + 101 + 101 + \ldots + 101 \]

We get \( 101 \cdot 100 / 2 = 5050 \)
Arc Length

SICILY

~20 mile intervals
235.73 miles

~10 mile intervals
287.18 miles
2.5 mile intervals
5 357.61 miles

10 miles
To measure length of curve:
* Divide $[a, b]$ into $n$ equal subintervals
* Add up the lengths of line segments shown.

* Take the limit, as $n \to \infty$.

For reasonable curves, this limit exists. For coastlines, it does not exist.
Let \( s(x_0) = \text{length of curve} \)
for \( a \leq x \leq x_0 \)

\[
\Delta s^2 = \Delta x^2 + \Delta y^2
\]

\[
\left( \frac{\Delta y}{\Delta x} \right)^2 = 1 + \left( \frac{\Delta y}{\Delta x} \right)^2
\]

Take limit as \( \Delta x \to 0 \)

\[
\left( \frac{ds}{dx} \right)^2 = 1 + \left( \frac{dy}{dx} \right)^2
\]

\[
\frac{ds}{dx} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2}
\]

\[
s(x_0) = \int_a^{x_0} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]
Basic arc length formula:

\[ s = \int_{a}^{b} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

\[ s = \text{length of curve shown}. \]

For most functions \( f(x) \), this integral is \textit{NASTY}.

\[ f(x) = x^3 \]

\[ a \to b \]

\[ \int_{a}^{b} \sqrt{1 + 9x^4} \, dx \]
Example: Find length of an ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

\(a \neq b\)

This leads to an unaccessible integral.
Another impossible but useful integral

\[
\int_a^b e^{-x^2} \, dx
\]

Example \( y = \sqrt{y^2 - x^2} \), \( 0 \leq x \leq m \)

Length is \( \frac{\pi m}{2} \)

\( s = \int_0^m \sqrt{1 + (y')^2} \, dx \)
\[ y' = \frac{-y}{\sqrt{m^2 - x^2}} \]
\[ y'^2 = 1 + \frac{x^2}{m^2 - x^2} = \frac{m^2 x^2 + x^2}{m^2 - x^2} = \frac{m^2}{m^2 - x^2} \]
\[ \sqrt{1 + (y')^2} = \frac{m}{\sqrt{m^2 - x^2}} \]
\[ S = \int_{0}^{\frac{\pi}{2}} m \frac{\max \theta}{\sqrt{m^2 - x^2}} \]
\[ = \int_{0}^{\frac{\pi}{2}} m^2 \sin \theta d\theta \]
\[ m \sqrt{x^2} = m \left[ \frac{\theta}{2} \right]_{0}^{\frac{\pi}{2}} = \frac{m \pi}{4} \]
\[ x = m \cos \theta \]
\[ dx = -m \sin \theta \, d\theta \]
\[ \sqrt{m^2 - x^2} = m \sin \theta \]
\[ \theta = \pi \]
\[ \theta = 0 \]
\[ = m \theta \bigg|_{0}^{\frac{\pi}{2}} = \frac{\pi m}{4} \]