Please submit a solution to your instructor by Friday, December 10 before 5 pm.

It is known that

\[
\frac{\sin \pi x}{\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)
= x \left(1 - x^2 \right) \left(1 - \frac{x^2}{4} \right) \left(1 - \frac{x^2}{9} \right) \left(1 - \frac{x^2}{16} \right) \cdots
\]  

(1)

The expression on the right is an infinite product, which needs to be interpreted as the limit of the corresponding finite products.

Notice that for a given value of \(x\), the value of the \(n\)th factor approaches 1 as \(n\) approaches infinity. Notice also that the infinite product vanishes whenever \(x\) is an integer; when \(x = \pm n\), the \(n\)th factor is zero. The expression on the left also vanishes whenever \(x\) is an integer, since the sine of any whole multiple of \(\pi\) is zero.

Both sides of the equation (1) have power series expansions. On the left, Taylor’s formula gives

\[
\sin \frac{\pi x}{\pi} = \sum_{i=0}^{\infty} \frac{\pi^{2i+1} x^{2i+1}}{(2i+1)!} 
= x - \frac{\pi^2 x^3}{3!} + \frac{\pi^4 x^5}{5!} - \frac{\pi^6 x^7}{7!} + \cdots
\]

The one on the right is more complicated. It is a power series in \(x\) in which the coefficients themselves are infinite series. The first two terms are

\[
x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)
= \quad x^3 \sum_{n=1}^{\infty} \frac{1}{n^2} + \cdots
= x^3 \sum_{n=1}^{\infty} \left(1 + \frac{1}{4} + \frac{1}{9} + \cdots \right) + \cdots
\]

You will need to find the next two terms (meaning the coefficients of \(x^5\) and \(x^7\)) yourself.

When two power series are equal, each coefficient of one of them is equal to the corresponding coefficient of the other one. Comparing the coefficients of \(x^3\) gives us the formula

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},
\]

which was originally proved by Leonhard Euler in 1735.

1. ((1/(N+1)) points) Use similar methods to show

\[
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.
\]