Example: \[ \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots \]

\[ a_n = \frac{2^n - 1}{2^n} \]

We say a sequence converges if \( \lim_{n \to \infty} a_n \) exists and is finite.

Recall \( \lim_{n \to \infty} a_n = L \) if for each \( \varepsilon > 0 \) there is an \( N \) such that
for \( n > N \), \(| L - a_n | < \varepsilon \).

In the above example,
\[ a_n = 1 - 2^{-n} \text{ and } \lim_{n \to \infty} a_n = 1 = L \]

\( \varepsilon = \frac{1}{25} = 0.04 \) isabella \( n = 2 \) will do

\( \varepsilon = \frac{1}{10000} = 0.0001 \) lilly

\( L - a_n = 1 - (1 - 2^{-n}) = 2^{-n} \)

\( L - a_n < 0.001 \) whenever \( n > 13 \).
Example of divergent sequence:

\[ a_n = (-1)^n n \]

\[ -1, 2, -3, 4, -5, 6, -7, 8, -9, \ldots \]

In this case, there is no limit.

Series: sum of infinitely many numbers

\[ b_1 + b_2 + b_3 + b_4 + b_5 + \ldots \]

E.g. \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \). \n
\( b_n = \) \( n \)th term is \( \frac{1}{2^n} \).

Associated with each series is a
Sequence of partial sums

\[ s_1, s_2, s_3, s_4, \ldots \]

where \( s_n = a_1 + b_2 + \ldots + b_n \) is the sum of the first \( n \) terms in the series.

**Definition.** The series \((s)\) converges (to \( L \)) if \( \lim_{n \to \infty} s_n = L \). The series diverges if the sequence does.

**Fact.** A series \( b_1 + b_2 + b_3 + \ldots \) does not converge unless \( \lim_{n \to \infty} b_n = 0 \).
An example where this fails

\[ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \ldots \]

\[ b_n = n \]

\[ a_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \]

Note: \( \lim_{n \to \infty} b_n = \infty \) Series diverges

and \( \lim_{n \to \infty} a_n = \infty \)

**WARNING**: A series can diverge even if its terms approach 0.

Example: Harmonic series
Why this series diverges

Compare it to another series

(A) \[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n} \]

Note each term in (A) is \( \geq \) the corresponding term in (B)

\[ \frac{1}{n} = b_n \]

Hence each partial sum in (A) is \( \geq \) the similar partial sum in (B)
The partial sums from (B) include $S_1 = 1$  $S_2 = 3/2$  $S_4 = 2$  $S_8 = 2\frac{1}{2}$  $S_{16} = 3$

$S_{32} = 3\frac{1}{2}$  $S_{64} = 4$  $S_{128} = 4\frac{1}{2}$  ...

Note since each term is $\geq 0$, $A_{n+1} > A_n$

We conclude $\lim_{n \to \infty} A_n = \infty$ and (B) diverges. Since $\frac{1}{n} \geq b_n$ for all $n$, (D) also diverges.

Thus, we have 2 series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$
with \( 0 \leq b_n \leq a_n \). Then:

- If \( \Sigma b_n \) diverges, so does \( \Sigma a_n \).
- If \( \Sigma a_n \) converges, so does \( \Sigma b_n \).

**WARNING:** If \( \Sigma a_n \) diverges, no conclusion about \( \Sigma b_n \) can be drawn.

- If \( \Sigma b_n \) converges, no conclusion about \( \Sigma a_n \) can be drawn.