Power series

\[ S = \sum_{n=0}^{\infty} c_n x^n \]

\( S \) is a variable

each \( c_n \) is a constant

Ratio test usually shows the series converges absolutely for \( |x| < M \)
and diverges for \( |x| > M \). Text inconclusive for \( x = \pm M \). \( M \)
is called the radius of convergence.

If series \( S \) converges to \( f(x) \)
for \( |x| < M \), then we can...
Differentiate or integrate term by term and get another power series with the same $x$ converging to $f'(x)$ or $\int f(x)dx$.

**Examples:**

1. $f(x) = \sum_{n=1}^{\infty} (\frac{1}{n+1}) x^n / n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

   converges for $|x| < 1$

2. $g(x) = \sum_{n=0}^{\infty} (\frac{1}{2n+1}) x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$

   converges for $|x| < 1$

**Differentiate:**

1. $1 - x + x^2 - x^3 + x^4 \ldots$
\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \Rightarrow \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^n \]

**Geometric Series**

with \( a = -1 \)

so \( f'(x) = \frac{1}{1 + x} \), \( f(x) = \int \frac{dx}{1 + x} = \ln(1 + x) + C \)

To determine the constant of integration \( C \), see what happens when \( x = 0 \):

\[ f(0) = \ln(1 + 0) + C = C \quad \text{and} \quad f'(0) = 0 \]

so \( C = 0 \)

Any series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots \) converges to \( \ln(1 + x) \) for \( |x| < 1 \).
Ratio test shows series diverges for $|x| > 1$. What about $x = \pm 1$?

For $x = -1$, series $= -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \ldots$

which diverges and $\ln(-1)$ is undefined.

For $x = 1$, series $= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$

which converges by alternating series test. If $f(1) = \ln 2 \approx 0.69$.

All bets are off for $x > 1$.

$x = 1.01$, then $\ln(1+x) = \ln(2.01) = 1$?

but series diverges.
Example 2 \( g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots \)

converges for \(|x| < 1\)

diverges for \(|x| \geq 1\)

\( g'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 \ldots \)

This is a geometric series with

ratio \( r = -x^2 \)

\( g'(x) = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1 - (-x^2)} = \frac{1}{1 + x^2} \) for \(|x| < 1\)

\( g(x) = \int \frac{dx}{1 + x^2} = \arctan(x) + C \)

Testing for \( x = 0 \) shows \( C = 0 \)
For $x = 1$, series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$.

It converges by alternating series test.

$arctan(1) = \pi / 4$

For $x = -1$, series is $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$.

It converges to $arctan(-1) = -\pi / 4$.

3. Series $\sum \limits_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = h(x)$

It converges for $|x| < 1$ to $\frac{1}{1-x}$

$h'(x) = \sum \limits_{n=0}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$h'(x) = 2(x+1 + 2x + 3x^2 + 4x^3 + \cdots)$

The derivative of $\frac{1}{1-x} = (1-x)^{-1}$ is 0
\[-(1-x)^{-2}(-1) = \left(\frac{1}{1-x}\right)^2 = h''(x)\]

\[h''(x) = 2 + 6x + 12x^2 + 20x^3 + \cdots\]

\[= \sum_{n=0}^{\infty} \binom{n+1}{n} x^\frac{n+1}{n} = \sum_{n=1}^{\infty} n(n+1)x^n = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)x^n}{n!}\]

**Derivative of** \[\frac{1}{(1-x)^2} \text{ is } \frac{2}{(1-x)^3}\]