WW 14.1 Find MacLaurin Series for \( \ln(1-x^2) \). We know \( \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \). Replace \( x \) by \(-x^2\) and get \( \ln(1-x^2) = \sum (-1)^{n+1} (-x^2)^n = -x^2 + \frac{x^4}{2} - \frac{x^6}{3} + \ldots \).
What are the limits?

\[
\lim_{n \to \infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}
\]

\[
\lim_{n \to \infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{\pi^3}{36}
\]

\[
\lim_{n \to \infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^5} = \frac{\pi^5}{450}
\]
Recall

Taylor series for \( f(x) \) cut at \( a \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

You choose \( a \). For \( a = 0 \) this is the MacLaurin series.

The Taylor polynomial is

\[
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k
\]

The remainder is

\[
R_n(x) = f(x) - T_n(x) = \text{with Taylor remainder}
\]
Taylor's inequality

Suppose we know:

\[ |f^{(n+1)}(x)| < M \quad \text{for} \quad |x-a| < d \]

Then:

\[ |R_n(x)| < \frac{M |x-a|^{n+1}}{(n+1)!} \quad \text{for} \quad |x-a| < d \]

Example: Consider \( f(x) = e^x \) for

\[ |x| < 0.1, \quad n = 3 \]
\[ T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \]

How close is this to \( e^x \)?

\[ f^{(n)}(x) = e^x \]

\( f(0) = e = 2.718 \)

\( f(-1) = \frac{1}{e} = 0.368 \ldots \)

In this interval, \( 0 < e^x < 2 \)

so we can take \( M = 2 \)
\[ |R_3(x)| \leq \left| \frac{M(x)^4}{4!} \right| \quad \text{for } |x| < 1 \]

\[ \leq \left| \frac{2 \times 10001}{24} \right| = \frac{100000833}{24} = 8.333 \times 10^{-6} \]

This means the difference between \( e^x \) and \( T_3(x) \) is smaller than \( 10^{-6} \).

Test this for \( x = -1 \)
\[ T_3(\eta) = 1 + \eta 1 + \frac{\eta(\eta - 1)}{2} + \frac{\eta(\eta - 1)(\eta - 2)}{6} \]

\[ = 1.105166667 \]

\[ C^{-1} = 1.105170918 \]

\[ \text{difference} \approx 4.252 \times 10^{-6} \]

For \( \eta = -1 \)

\[ T_3(-1) = 1 - 1 + \frac{0!}{2} - \frac{0!}{6} \]

\[ = 1 - 0.483333318 \]
\[ c^{-1} = 0.4837418 \]

difference = \( 4.08 \times 10^{-6} \)

\[ g(x) = x^m \quad x = x - \frac{x^3}{3!} \quad n = 4 \]
\[ d = 1 \]

\[ T_3(x) = x - \frac{x^3}{3!} \]

\[ f^{(5)}(x) = \cos \]

\[ |f^{(5)}(x)| \leq M \]

Estimate \( \sin x \) using Taylor's inequality.
\[ R_n(x) < \left| \frac{M(x-a)^5}{5!} \right| = \frac{0.00001}{120} \]

\[ = 8.33 \times 10^{-8} \]

\[ x = n \]

\[ T_n(\lambda) = n! - 0.001 \approx 0.9983333 \]

\[ \sin(\lambda) = 0.998334166 \]

\[ \text{diff} = 8.33 \times 10^{-8} \]