More on partial fractions. Use them for rational functions \( \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) are polynomials.

1. Make sure \( \deg(p(x)) \) (degree means largest exponent) is less than \( \deg(q(x)) \). If not, divide \( q(x) \) into \( p(x) \) and get a quotient \( q'(x) \) and remainder \( r(x) \). This
\[ \frac{p(x)}{q(x)} = \frac{q(x) \cdot s(x) + m(x)}{q(x)} = s(x) + \frac{m(x)}{q(x)} \]

This means \( \frac{p(x)}{q(x)} \). 

\[ w(x) = \deg m(x) < \deg q(x) \]

\( q(x) \) is a polynomial.

1. Factor the denominator \( q(x) \).

There are two possible wrinkles:

(i) \( q(x) \) may have repeated factors.

(ii) \( q(x) \) has reducible quadratic factors.
Example of \( i \):

\[
\frac{x^2+1}{(x-1)^3} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}
\]

\(A, B, C\) are constants

Text book method

RHS = \[
\frac{A}{(x-1)^3} + \frac{B(x-1)}{(x-1)^3} + \frac{C(x-1)^2}{(x-1)^3}
\]

= \[
\frac{A + Bx - B + Cx^2 - 2C}{(x-1)^3}
\]

(1)

\(c = 1\)

\(b - 2c = 0\)

\(a - b + c = 1\)

ENJOY!

\[
= \frac{C(x^2 + (B-2c)x + (A-B+C)}{(x-1)^3} = \frac{x^2+1}{(x-1)^3}
\]
Heaviside's method

\[
\frac{x^2+1}{(x-1)^3} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}
\]

Multiply both sides by \((x-1)^3\)

\[
x^2+1 = A + (x-1)(B + C)
\]

\[
\text{let } x = 1, \text{ i.e. } x = 1
\]

\[
2 = A
\]

\[
\left(\frac{x^2+1}{(x-1)^3}\right) - \frac{2}{(x-1)^3} = \frac{B}{(x-1)^2} + \frac{C}{x-1}
\]

\[
B = \frac{x^2-1}{(x-1)^3} = \frac{(x-1)(x+1)}{(x-1)^3} = \frac{x+1}{(x-1)^2}
\]
now reads
\[
\frac{x+1}{(x-1)^2} = \frac{B}{(x-1)^2} + \frac{C}{(x-1)}.
\]

And this before etc.

Example of (ii), irreducible quadratic factors in \( q(x) \):
\[
\frac{x^2 + x + 1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C + D}{x^2+1}
\]
\[
= \frac{x^2 + x + 1}{x^4 - 1}.
\]

To integrate the third term:
\[
\int \frac{C + D}{x^2 + 1} \, dx = C \int \frac{x \, dx}{x^2 + 1} + D \int \frac{dx}{x^2 + 1}
\]
\[
m = x^2 + 1
\]
\[
dn = 2x \, dx
\]
\[
x \, dx = \frac{dn}{2}.
\]
\[ \frac{x^2 + x + 1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx + D}{x^2+1} \]

**Textbook method to find A, B, C, D**

\[ \text{RHS} = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2+1) \]

\[ = A(x^3 + x^2 + x + 1) + B(x^2 - x^2 - x - 1) + (Cx + D)(x^2 + 1) \]

\[ = A + B + C \cdot x^3 + (A - B + D) \cdot x^2 + (A + B - C) \cdot x + (A - B - D) \]

\[ A + B + C = 0 \quad \text{4 equations} \]
\[ A - B + D = 1 \quad \text{4 unknowns} \]

\[ \text{YUCK} \]
\[ A - B - D = 1 \]

\[ \frac{x^2 + x + 1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx + D}{x^2 + 1} \]

Heaviside's method. Find A and B, subtract those terms + look for a miracle.

To find A: multiply by \( x-1 \)

\[ \frac{x^2 + x + 1}{(x+1)(x^2+1)} \]

\[ \frac{3}{4} = A \]
\[
\frac{x^2 + x + 1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}
\]

To find \(B\), multiply by \(x+1\):

\[
\frac{x^2 + x + 1}{(x-1)(x^2+1)} = B + \frac{(x+1)(-2)}{(x+1)}
\]

Let \(x+1=0\), i.e., \(x = -1\),

\[
-\frac{1}{4} = \frac{1-1+1}{(-2) + 2}
\]

\[
\frac{x^2 + x + 1}{x^4 - 1} - \frac{3}{4(x-1)} + \frac{1}{4(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C(x+D)}{x^2+1}
\]

\[
A = \frac{3}{4}
\]

\[
\frac{x^2 + x + 1}{x^4 - 1} = \frac{1}{4(x-1)} + \frac{1}{4(x+1)} + \frac{\frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x^2+1}
\]
\[
= \frac{4x^2 + 4x + 4 - 3(x^1)(x^3) + (x-1)(x^2)}{4(x^4-1)}
\]
\[
= \frac{4x^2 + 4x + 4 - 3(x^3 + x^2 + x + 1) + (x^2 - x^3 + x - 1)}{4(x^4-1)}
\]
\[
= \frac{-2x^3 + 2x}{4(x^4-1)} = \frac{-2x(x^2-1)}{4(x^2+1)(x^2-1)} = \frac{-x}{2(x^2+1)} \left( \text{MIRACLE!} \right)
\]

Hence \( C = -1/2 \) and \( D = 0 \)

\[
= \frac{x^2 + x + l}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C+D}{x^2+1}
\]
\[
\int \frac{3}{x^4 - 1} + \frac{x/2}{x^2 + 1} \mathrm{d}x
\]

We can integrate this!