Integral test

We need to assume 2 things about the series \( \sum_{n=1}^{\infty} a_n \):

1. All terms are non-negative, \( a_n \geq 0 \).
2. The terms are non-increasing, \( a_{n+1} \leq a_n \).

If \( f(x) \) diverges, so does the series.
Suppose there is a nonincreasing continuous function \( f(x) \) with \( f(n) = a_n \).

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

It satisfies our conditions since

\[ a_n > a_{n+1} > 0 \quad \text{i.e.} \quad \frac{1}{n} > \frac{1}{n+1} > 0, \]

\( f(x) = \frac{1}{x} \) is the desired continuous function.
\[ A_n = a_1 + \ldots + a_n \geq \int_1^n \frac{dx}{x} \]
Hence \( \lim_{n \to \infty} a_n \geq \lim_{n \to \infty} \int_1^n \frac{1}{x} \, dx = \lim_{n \to \infty} \ln(n) = \infty \)

Conclusion: Series diverges.

The integral test can also be used to prove that a series converges.

We need a function \( f(x) \) as above with the same conditions as before.
Example: \[ \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots, \]

\[ f(x) = \frac{1}{x^2} \]

We know:

\[ \lim_{t \to \infty} \int_{1}^{t} \frac{dx}{x^2} = \lim_{t \to \infty} \left[ -\frac{1}{x} \right]_{1}^{t} = \lim_{t \to \infty} \left( -\frac{1}{t} - (-1) \right) = 1 \]

Integral converges
\[ a_1^7 a_2 + a_3 + a_4 + a_5 < 1 + \int_1^\infty \frac{dx}{x^2} \]
\[ a_n = a_1 + \cdots + a_n < 1 + \sum_{k=1}^{\infty} \frac{k}{k^2} \quad \text{for any } n > 1 \]

\[ \lim_{n \to \infty} S_n < 1 + \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{k^2} = 1 + 1 = 2 \]

\textbf{Conclusion:} This series converges.

But to what? ? ? ?
Then (Euler 1730) \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \).

He gave a formula for \( \sum_{n=1}^{\infty} \frac{1}{n^k} \) for any positive integer \( k \).

What about \( \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots \)?
Integral test shows that it converges.
Its value is a mystery to this day.
General statement:

If we have series \( \sum_{n=1}^{\infty} a_n \) with \( 0 \leq a_{n+1} \leq a_n \) for all \( n \) and a continuous non-increasing function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(a_n) = a_n \), then series converges \( \iff \int_0^\infty f(x) \, dx \) converges.
series diverges \rightarrow integral diverges