Recall the Maclaurin series for a smooth function $f(x)$ is

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots
$$

A variation: The Taylor series at $a$:

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
$$

For $a=0$ this is the Maclaurin series.
\[ \pi \left( \frac{1}{2} \right) \cdot \pi \left( \frac{3}{2} \right) \cdot \pi \left( \frac{5}{2} \right) \cdots \]

We use the ratio test to see where this converges. We usually find it converges for \( |x-a| < r \)

\[ r = \text{radius of convergence} \]

It could be infinity.

\[ \text{Thm 5} \quad \text{If there is a series of the form } \sum_{n=0}^{\infty} c_n (x-a)^n \]
converging to \( f(x) \) for small \( |x-a| \), then it is the Taylor series.

A bad example: A smooth function without a series converging to it.

\[
\begin{align*}
  f(x) &= e^{-1/x^2} \\
  f(0) &= \lim_{x \to 0} e^{-1/x^2} = 0 \\
  f'(x) &= \left( \frac{2}{x^3} \right) e^{-1/x^2}
\end{align*}
\]
\[ f'(0) = \lim_{x \to 0} f'(x) = 0 \]

It turns out that \( f^{(n)}(0) = 0 \) for all \( n \).

This means the Maclaurin series is 0, \( \lim f(x) \neq 0 \).
Quantities related to the Taylor (or Maclaurin) series:

\[ T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i = \text{n-th partial sum} \]

= a number we can calculate in a finite amount of time.
\[ R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} (x-x_0)^k \] is the Taylor remainder. We want this to be small.

Suppose you want to know \( \log 2 \)
\[ \log 2 = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{b!} + \ldots \]
Suppose we want 8 digit accuracy
We want $F(c) = 10.8$.

Question: What should $m$ be?

$m = 32.2$ kg.

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\[ |f^{(n+1)}(x)| < M \quad \text{for } |x-a| \leq d \]

Then
\[ |R_n(x)| \leq \frac{M |x-a|^{n+1}}{(n+1)!} \quad \text{for } |x-a| \leq d \]
We will use this to find $\sin 2$ to 8 digit accuracy.

Use Maclaurin series for $a = 0$.

Each derivative of $f(x) = \cos x$ is either $\pm \cos x$ or $\pm \sin x$.

So $|f^{(n)}(x)| \leq 1$ for any $x$. 

and any \( n \). Hence \( M = 1 \), Taylor's punch line for
\[
a = 0, \quad d = \sqrt{2} = x, \quad x-a = \sqrt{2}
\]
\[
\left| R_n(x) \right| \leq \frac{1}{(n+1)!} \left( \sqrt{2} \right)^{n+1}
\]
We want this to be \( \leq 10^{-8.5} \).
\[ n = 6 \quad \rightarrow \quad |R_6(n, 2)| \leq \frac{(2^2)^7}{7!} = 2.54 \times 10^{-9} \]

\[ \omega_0(n, 2) = 1.9800665778 \]

Calculator

Series answer:

\[ = 1 - \frac{(12)^2}{24} + \frac{(12)^4}{720} - \frac{(12)^6}{720} \]

\[ = 0.9800665778 \]
Taylor’s estimate for $n=7$

$$|R_n(x)| < \frac{(4.2)^8}{8!} = 6.35 \times 10^{-11}$$