Example 3 \[ y = x^2 \quad 0 \leq x \leq a \quad x^2 = y \leq a^2 \]

Rotate about y-axis. The solid is a paraboloid. Find its volume. Rotating a vertical strip about the y-axis gives a shell.
Volume of shell is area of strip \( \times \) perimeter of circle

\[
\int \frac{\pi}{2} \int (a^2 - x^2) \, dx \, \pi x
\]

Volume of solid

\[
= \int_0^a \text{shell volume} = \int_0^a 2\pi x (a^2 - x^2) \, dx
\]

\[
= 2\pi \left[ \frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a
\]

\[
= 2\pi \left( \frac{a^4}{2} - \frac{a^4}{4} \right)
\]
Again, we have the region bounded by $0 \leq x \leq a$ and $x^2 \leq y \leq a^2$.

Rotate the region about the $y$-axis and find the volume.

Divide the region into horizontal slices and integrate with respect to $y$. 

\[
= 2\pi \left( \frac{a^4}{2} - \frac{a^4}{4} - 0 \right) = 2\pi \frac{a^4}{4} = \frac{\pi a^4}{2}
\]
Rotating a horizontal strip gives a horizontal slice.

The volume is \( dV = \text{area} \times \text{thickness} \)

\[ \pi x^2 \cdot dy \]

\[ 0 \leq y \leq a^2 \quad \text{and} \quad x = \sqrt{y} \quad \Rightarrow \quad x^2 = y \]

Volume:

\[ \int_0^{a^2} \pi x^2 \, dy = \int_0^{a^2} \pi y \, dy = \left. \frac{\pi y^2}{2} \right|_0^{a^2} = \frac{\pi a^4}{2} \]
When rotating a region about a vertical line (e.g., the y-axis) there are 2 choices:

1. Divide region into vertical strips and integrate with respect to $x$. **Shell Method**

2. Divide region into horizontal strips and integrate w.r.t. $y$. **Washer Method**
Similar discussion for rotating about a horizontal line, e.g. the $x$-axis.

WW 2.6 The region between the graphs of $y = x^2$ and $y = 6x$ is rotated about the line $y = 36$. Find the volume.

ANNEAUL RING

Washer in the region between 2 circles (annulus)
radius of larger circle is \[ 36 - x^2 = r_1 \]
smaller radius is \[ 36 - 6x = r_2 \]
area of annulus is \[ \pi M_1^2 - \pi M_2^2 \]
\[ = \pi [(36 - x^2)^2 - (36 - 6x)^2] \]
\[ = \pi [(1296 - 72x^2 + x^4) - (1296 - 432x + 36x^2)] \]
\[ = \pi (x^4 - 108x^2 + 432x) \]
Volume of a washer = area \times thickness

= \pi (x^4 - 108x^2 + 132x) \, dx

V = \int_0^5 \pi (x^4 - 108x^2 + 132x) \, dx = \text{some number}

WW 2.8. The region enclosed by

x = 0, x = 1, y = 0 and y = x^2 + 3

in rotated about x-axis. Find the volume.
Vertical strips + washer method.

Area of slice: \( A = \pi \, r^2 = \pi \left( x^3 + 3 \right)^2 \)

\( r = \text{radius} \) \( = \pi \left( x^4 + 6x^2 + 9 \right) \)

Volume:

\[ V = \int_0^1 \pi \, r^2 \, dx = \pi \int_0^1 \left( x^4 + 6x^2 + 9 \right) \, dx \]

\[ = \pi \left( \frac{x^5}{5} + 6 \frac{x^3}{3} + 9x \right)_0^1 \]

\[ = \pi \left( \frac{1}{5} + 6 \frac{1}{3} + 9 \right) \]
\[
\pi \left( \frac{1}{15} \cdot \frac{3}{4} \cdot 9 - 0 \right) = \pi \left( \frac{44}{60} + \frac{45}{60} + 9 \right)
\]
\[
= \pi \left( 9 \cdot \frac{49}{60} \right)
\]

WW2.

Region bounded by \( y^2 = x^3 \) and \( y = 1 \) is rotated about \( y = 4 \). Find volume.
Vertical strip: shells

area of shell = \pi (2y)^2 dy

Volume of solid = \int_{y=0}^{y=1} \pi (2y)^2 dy

= \pi \int_{y=0}^{y=1} 4y^2 dy

= \pi \left[ \frac{4y^3}{3} \right]_{y=0}^{y=1}

= \frac{4\pi}{3}

Horizontal strip: shells

length of strip is 2x = 2\sqrt{y} = h

radius of shell = 4 - y = r

Volume of shell = 2\pi rh dy

= 2\pi (4 - y) 2\sqrt{y} = 16\pi y^{1/8} - 4\pi y^{9/8}

Volume of solid = \int_{y=0}^{y=1} \left( 16\pi y^{1/8} - 4\pi y^{9/8} \right) dy
\[
\begin{align*}
\text{II} & = \frac{16\pi}{9} \frac{9/8}{9/8} - \frac{4\pi}{17/8} \frac{17/8}{0} \\
& = \left( \frac{128}{\pi} \frac{9/8}{9} - \frac{32\pi}{17} \frac{17/8}{0} \right) \left( \frac{1}{0} \right) \\
& = \frac{128\pi}{9} - \frac{32\pi}{17} - 0 = \pi \left( \frac{\text{something}}{15/3} \right)
\end{align*}
\]