Fundamental Theorem of Calculus

Let \( f(x) \) be a continuous function for \( a \leq x \leq b \). There is a differentiable function \( F(x) \) as shown below with \( F'(x) = f(x) \).
Corollary: To find the area under the curve, find a function $g(x)$ with $y = f(x)$.
\[ g'(x) = f(x). \] Then the area is
\[ g(b) - g(a) = \int_a^b f(x) \, dx \]

\( g(x) \) is called an antiderivative of \( f \).

**Example 1**

\[ \int_{-1}^{1} \frac{dx}{x^2} = ? \]

\[ f(x) = \frac{1}{x^2} \]
\[ g(x) = -\frac{1}{x} \]

BAD: \( f(x) \) not defined at \( x = 0 \)
\[ \int_{-1}^{1} \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_{-1}^{1} = (-1) - (1) = -2 \]
Example 2 \[ \int_1^5 e^x \, dx = e^x \bigg|_1^5 \]

\[ f(x) = e^x \]

\[ g(x) = e^x \]

\[ = e^5 - e^1 = e^5 - e \]

This is the area under the curve.
Example 3

\[ \int_0^1 x^3 \, dx = \left. \frac{x^4}{4} \right|_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4} \]

Example 4

\[ \int_2^4 \frac{dx}{x} = \left. \ln(x) \right|_2^4 = \ln(4) - \ln(2) \]

\[ \ln(4/2) = \ln 2. \]
A method (one of many) for finding the anti-derivatives:

**Substitution Rule**

\[
\int_a^b h(g(x)) g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
\]

where \( u = g(x) \)

This is related to the chain rule.
Example 5

\[ \int_0^1 x^3 \sin(x^4 + 3) \, dx \]

\[ = \frac{1}{4} \int_0^4 g'(x) \sin(g(x)) \, dx = \frac{1}{4} \int_0^4 \sin(u) \, du \]

\[ = \frac{1}{4} \left[ \cos(u) \right]_0^4 = \frac{\cos 4 - \cos 0}{4} \]
\[ \int x^3 \sin (x^4 + 3) \, dx = \frac{1}{4} \int - \cos (x^4 + 3) \, d(x^4 + 3) + C \]

We can check this by differentiating.

Let \( y(x) = -\frac{1}{4} \cos (x^4 + 3) \)

**USE CHAIN RULE**

\[ y(x) = \frac{1}{4} \cos (x^4 + 3) \]

\[
\frac{dh}{dx} = \frac{dh}{du} \frac{du}{dx} = \left(\frac{\sin u}{4}\right) 4x^3 = (\sin u) x^3
\]

\[
= x^3 \sin (x^4 + 3)
\]

**Example 6**

\[
\int \sqrt{3x+1} \, dx
\]

\[
= \frac{1}{3} \int 3 \sqrt{3x+1} \, dx
\]

\[
g(u) = 3x+1 \quad \frac{du}{dx} = 3 \quad \theta(u) = \sqrt{u}
\]
\[ \int \frac{1}{\sqrt{n}} \, dn = \frac{1}{2} \int \sqrt{n} \, dn = \frac{1}{3} \sqrt{n^3} \, dn \]

\[ \frac{1}{3} \left( \frac{m^{3/2}}{3^{1/2}} + C \right) = \frac{2}{9} \frac{m^{3/2}}{C} = \frac{2}{9} \left( 3x^4 + 1 \right)^{3/2} + C \]