Part A
1. (10 points)

A circular swimming pool has a diameter of 24 ft., the sides are 5 ft. high, and the depth of the water is 4 ft. How much work is required to pump all of the water out over the side? (Use the fact that water weighs 62.5 lb/ft\(^3\).)

Solution:

Let \( y \) be the vertical distance from the top of the pool. In this problem it ranges from 1 to 5. The work required to lift the layer of water at distance \( y \) over the top is \( W = F \cdot y \Delta y \), where \( \Delta y \) is the thickness of the layer. Then \( F = 62.5 \cdot 5 \cdot V \), where \( V \) is the volume of the layer, which is \( \pi \cdot 12^2 \cdot \Delta y \). Now the work required to pump all the water out is given by:

\[
W = \int_{1}^{5} \pi \cdot 12^2 \cdot 62.5 \cdot y \, dy \\
= \pi \cdot 12^2 \cdot 62.5 \cdot \frac{y^2}{2}\bigg|_{1}^{5} \\
= \pi \cdot 12^2 \cdot 62.5 \cdot \left(\frac{25}{2} - \frac{1}{2}\right) \\
= \pi \cdot 12^3 \cdot 62.5 \, \text{ft-lb} \\
= 108,000 \pi \, \text{ft-lb}
\]

2. (10 points)

Find the definite integral

\[
\int_{0}^{\frac{\pi}{2}} x \cos(2x) \, dx
\]

Solution: We use integration by parts with

\[
\begin{align*}
    u &= x & dv &= \cos(2x) \, dx \\
    du &= dx & v &= \frac{\sin(2x)}{2}
\end{align*}
\]
so

\[
\int_0^\pi x \cos(2x) \, dx = \left. \frac{x \sin(2x)}{2} \right|_0^\pi - \int_0^\pi \frac{\sin(2x)}{2} \, dx
\]

= \left. 0 - \frac{\cos(2x)}{4} \right|_0^\pi

= \frac{1}{2}
3. (10 points)

Solve this integral:

\[ \int \frac{\sqrt{9 - x^2}}{x^2} \, dx \]

We use the substitution \( x = 3 \sin \theta \), so that \( dx = 3 \cos \theta d\theta \) and \( \sqrt{9 - x^2} = 3 \cos \theta \). Then

\[
\int \frac{\sqrt{9 - x^2}}{x^2} \, dx = \int \frac{3 \cos \theta}{9 \sin^2 \theta} \cdot 3 \cos \theta d\theta = \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta = \int \cot^2 \theta d\theta = \int (\csc^2 \theta - 1) d\theta = -\cot \theta - \theta + C
\]

Drawing a triangle, we see that \(-\cot \theta - \theta + C\) reduces to

\[-\frac{\sqrt{9 - x^2}}{x} - \arcsin(x/3) + C\]

4. (10 points)

Evaluate this integral:

\[ \int \frac{1}{x^2 + x} \, dx \]

We use partial fractions:

\[
\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}
\]

Adding the fractions on the right side of the equation and comparing numerators we obtain:

\[ 1 = A(x + 1) + Bx, \]

and it follows that \( A = 1 \) an \( B = -1 \). So the integral becomes

\[
\int \left( \frac{1}{x} - \frac{1}{x+1} \right) \, dx = (\ln |x| - \ln |x + 1|) + C
\]
5. (10 points)

(a) Does the series \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \) converge or diverge? Why?

(b) Does the series \( \sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4} \) converge or diverge? Why?

**Solution (a):** The function \( f(x) = \frac{\ln x}{x} \) is positive, continuous and decreasing (look at derivative!) for \( x > 1 \) Thus we can apply the integral test

\[
\int_{1}^{\infty} \frac{\ln x}{x} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{\ln x}{x} \, dx = \lim_{t \to \infty} \left[ \frac{(\ln x)^2}{2} \right]_{1}^{t} = \lim_{t \to \infty} \frac{(\ln t)^2}{2} = \infty
\]

Since this improper integral is divergent, the series is also divergent.

**Solution (b):** The general term of the series is \( a_n = \frac{n^2}{5n^2 + 4} \). Then:

\[
\lim_{n \to \infty} \frac{n^2}{5n^2 + 4} = \lim_{n \to \infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0
\]

Thus, the series diverges by the Divergence Test.

6. (10 points) Find all \( x \) for which the following power series converges, i.e. find the interval of convergence:

\[
\sum_{n=1}^{\infty} \frac{-1^n}{n+1} (x + 1)^n
\]

**Solution:** We use the Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{-1^{n+1}}{n+2} (x + 1)^{n+1} \right| = \frac{1}{n+2} \lim_{n \to \infty} |x + 1| \frac{n+1}{n+2}
\]

\[
= |x + 1| \lim_{n \to \infty} \frac{1 + 1/n}{1 + 2/n}
\]

\[
= |x + 1|
\]

Since the series converges for \(|x + 1| < 1\), the radius of convergence is 1. We still have to test the endpoints:

When \( x = 0 \), the series becomes

\[
\sum_{n=1}^{\infty} \frac{-1^n}{n+1}
\]
which converges by the Alternating Series Test. When $x = 2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{-1^n}{n+1} \cdot (-2)^n$$

This is a divergent series. You can see this by the Divergence Test or by limit comparison to a harmonic test.

Therefore, the interval of convergence for this series is $(-2, 0]$.

Part B
7. (10 points)

The power series for $e^{-x^2}$ is given by

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} + \cdots$$

Use this to get the power series for the area under the bell curve,

$$f(x) = \int_{0}^{x} e^{-t^2} \, dt.$$  

You can either use summation notation or write down the first 5 non-zero terms.

Solution: Integrating the given series term by term gives

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} + \cdots$$
8. (10 points)

(a) Find $T_3(x)$, the third degree Taylor polynomial for $f(x) = \sqrt{x}$ at $x = 4$.

(b) Use Taylor’s inequality to find the largest integer $k$ such that the error when $T_3(x)$ is used as an approximation for $f(x)$ on the interval $4 \leq x \leq 5$ is less than $10^{-k}$.

Solution (a): We should first compute four derivatives and their values at $a = 1$:

\[
\begin{align*}
    f(x) &= x^{1/2} \quad f(4) = 2 \\
    f'(x) &= \frac{1}{2}x^{-1/2} \quad f'(4) = \frac{1}{4} \\
    f''(x) &= -\frac{1}{4}x^{-3/2} \quad f''(4) = -\frac{1}{32} \\
    f'''(x) &= \frac{3}{8}x^{-5/2} \quad f'''(4) = \frac{3}{256} \\
    f^{(4)}(x) &= -\frac{15}{16}x^{-7/2} \quad f^{(4)}(4) = -\frac{15}{2024}
\end{align*}
\]

Thus

\[
T_3(x) = 2 + \frac{x - 4}{4} - \frac{(x - 4)^2}{32 \cdot 2!} + \frac{3(x - 4)^3}{256 \cdot 3!} - \frac{15(x - 4)^4}{512}
\]

Solution (b): Notice that $4 \leq x \leq 5$ means that $|x - 4| \leq 1$. Also, since $|f^{(4)}(x)| = \frac{15}{16x^{7/2}}$, we know that

\[
|f^{(4)}(x)| \leq \frac{15}{16 \cdot 4^{7/2}} = \frac{15}{1024},
\]

on the interval $4 \leq x \leq 5$. Thus, by Taylor’s Inequality, we have:

\[
|R_3(x)| \leq \frac{15}{1024 \cdot 4} \cdot |x - 1|^4
\]

\[
\leq \frac{15}{24 \cdot 1024}
\]

\[
< 10^{-3}
\]
9. (10 points) (a) Write the general formula for the Taylor series of a function \( f(x) \) at \( a \) (or “about \( a \)” or “centered at \( a \)”).

(b) Write the Taylor series of \( f(x) = e^{2x} \) at \( a = 1 \). You can either use summation notation or write down the first 5 non-zero terms.

Solution: (a) The general formula is

\[
\sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}
\]

(b) For \( f(x) = e^{2x} \), the \( n \)th derivative is \( f^{(n)}(x) = 2^n e^{2x} \), so \( f^{(n)}(1) = 2^n e^2 \) and the series is

\[
\sum_{n=0}^{\infty} f^{(n)}(1) \frac{(x-1)^n}{n!} = e^2 \sum_{n=0}^{\infty} \frac{2^n(x-1)^n}{n!}
\]

\[
= e^2 + 2e^2(x-1) + \frac{4e^2(x-1)^2}{2!} + \frac{8e^2(x-1)^3}{3!} + \frac{16e^2(x-1)^4}{4!} + \ldots
\]

\[
= e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4e^2(x-1)^3}{3} + \frac{2e^2(x-1)^4}{3} + \ldots
\]

10. (10 points) Consider the cycloid defined by the parametric equations

\[
x = 2(t - \sin t) \quad \text{and} \quad y = 2(1 - \cos t)
\]

for \( 0 \leq t \leq 2\pi \).

(a) For which values of \( t \) is the tangent line vertical? Find the corresponding points.

(b) For which values of \( t \) is the tangent line horizontal? Find the corresponding points.

Solution: We have

\[
\frac{dx}{dt} = 2(1 - \cos t) \quad \text{and} \quad \frac{dy}{dt} = 2 \sin t
\]

(a) The tangent line is vertical when \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} \neq 0 \), i.e. when \( \cos t = 1 \), which means \( t = 0 \) or \( 2\pi \) so \((x, y) = (0, 0)\) or \((4\pi, 0)\).

(b) It is horizontal when \( \frac{dy}{dt} = 0 \) and \( \frac{dx}{dt} \neq 0 \), i.e. when \( \sin t = 0 \) but \( \cos t \neq 1 \). This happens when \( t = \pi \) and \((x, y) = (2\pi, 4)\).
11. **(10 points)** Find the length of the cycloid of the previous problem for $0 \leq t \leq \pi$.

*Hint:* Use the half angle formula $\sin(\theta/2) = \sqrt{(1 - \cos \theta)/2}$.

**Solution:**

We have

$$
\left( \frac{dx}{dt} \right)^2 = (2(1 - \cos t))^2 = 4 - 8 \cos t + 4 \cos^2 t
$$

and

$$
\left( \frac{dy}{dt} \right)^2 = (2 \sin t)^2 = 4 \sin^2 y
$$

so

$$
\sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} = \sqrt{4 - 8 \cos t + 4 \cos^2 t + 4 \sin^2 t}
$$

$$
= \sqrt{8 - 8 \cos t}
$$

$$
= 4 \sqrt{(1 - \cos t)/2}
$$

$$
= 4 \sin(t/2).
$$

It follows that the arc length is

$$
L = 4 \int_0^\pi \sin(t/2) dt
$$

$$
= 8 \int_0^{\pi/2} \sin(u) du \quad \text{where } u = t/2 \text{ and } dt = 2 du
$$

$$
= 8.
$$
12. (10 points)

Find the area of the surface obtained rotating the semicircle \( y = \sqrt{25 - x^2}, 3 \leq x \leq 4, \) about the x-axis.

**Solution:** Let \( f(x) = \sqrt{25 - x^2}, \) so

\[
\begin{align*}
  f'(x) &= \frac{x}{\sqrt{25 - x^2}} \\
  1 + f'(x)^2 &= 1 + \frac{x^2}{25 - x^2} = \frac{25}{25 - x^2} \\
  \sqrt{1 + f'(x)^2} &= \frac{5}{\sqrt{25 - x^2}}
\end{align*}
\]

Then the surface area is

\[
S = \int_{3}^{4} 2\pi f(x)\sqrt{1 + f'(x)^2} \, dx
\]

\[
= \int_{3}^{4} 2\pi 25 - x^2 \frac{5}{\sqrt{25 - x^2}} dx
\]

\[
= 10\pi \int_{3}^{4} dx
\]

\[
= 10\pi.
\]

13. (10 points)

Find the area enclosed by the 8-leafed rose defined by \( r = \sin 4\theta \) for \( 0 \leq \theta \leq 2\pi. \)

**Solution:** Using the area formula for polar curves, we get

\[
A = \frac{1}{2} \int_{0}^{2\pi} \sin^2 4\theta d\theta
\]

\[
= \frac{1}{2} \int_{0}^{2\pi} \frac{1 - \cos 8\theta}{2} d\theta
\]

\[
= \frac{1}{32} \int_{0}^{16\pi} \frac{1 - \cos u}{2} du \quad \text{where } u = 8\theta \text{ and } d\theta = du/8
\]

\[
= \frac{\pi}{2}.
\]