More examples of power series.

1. Find the Taylor series for \( \sin x \) at \( a = \pi/2 \).

\[
\begin{align*}
    f(x) & = \sin x \\
    f'(x) & = \cos x \\
    f''(x) & = -\sin x \\
    f'''(x) & = -\cos x \\
    f^{(4)}(x) & = \sin x \\
    \vdots
\end{align*}
\]

\[
\begin{align*}
    \sin (\pi/2) & = 1 \\
    \cos (\pi/2) & = 0 \\
    -\sin (\pi/2) & = -1 \\
    -\cos (\pi/2) & = 0 \\
    \sin (\pi/2) & = 1 \\
    \vdots
\end{align*}
\]
The series is

\[ 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \ldots \]

Recall the Maclaurin for \( \cos x \) is

\[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \]

is also the series for \( \cos (x-\pi/2) \)

A trig identity says \( \cos (x-\pi/2) = -\sin x \)

Use a series to estimate

\[ \int_0^\infty e^{-x^4} \, dx \] to 8 decimal places.
The Maclaurin series for $e^x$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

Replace $x$ by $-3x^9$ and get

$$e^{-3x^9} = \sum_{n=0}^{\infty} \frac{(-3x^9)^n}{n!} = 1 - \frac{3x^9}{2} + \frac{9x^{18}}{6} - \frac{27x^{27}}{2} + \ldots$$

Integrating term-by-term gives

$$\int_0^x e^{-3x^9} \, dx = x - \frac{3x^{10}}{5} + \frac{9x^{19}}{2 \cdot 9} - \frac{27x^{28}}{6 \cdot 13} + \ldots$$
\[ = \sqrt{2} - \frac{3}{5} (\sqrt{2})^5 + \frac{9}{2} (\sqrt{2})^9 - \frac{9}{26} (\sqrt{2})^{13} + \cdots \]

\[ = \sqrt{2} - (1.92 \times 10^{-4}) + (2.56 \times 10^{-7}) - (3.83 \times 10^{-10}) \]

This is an alternating series. Its sum is between any 2 successive partial sums. \( S_1 = \sqrt{2} \), \( S_3 \) and \( S_4 \) differ by \( 2.83 \times 10^{-10} \). The first 3 terms give us 9 digit accuracy.
(3) Find $e^{\alpha^2}$

Calculator says $e^{\alpha^2} = 1.221402758$

The series we need is

$\sum x^n/n! = 1 + x + x^{3/2} + \frac{x^6}{6} + \ldots$

This is not alternating.

We need to use Taylor's inequality.

We need to ensure that $|R_1(x)| \leq 5 \times 10^{-10}$ to get 9 digit accuracy.
\[ f^{(m+1)}(x) = e^x \text{ for all } m. \]
\[ e^{12} < 2 \text{ since } e^6 = 2 \]
so we can take \( M = 2 \). We need
\[ \frac{2 (\lambda 2)^m}{(m+1)!} < 5 \times 10^{-10} \]
\[ m = 5 : \quad \frac{2 (\lambda 2)^6}{6!} = 1.7777\ldots \times 10^{-7} \]
\[ m = 6 : \quad \frac{2 (\lambda 2)^7}{7!} = 5.079 \times 10^{-9} \]
\[ (\lambda 28)/8! \approx 6.3 \times 10^{-10} \]
\[ n = 7 \qquad \frac{2^{(112)}}{8^1} = 1.26 \times 10^{-10} \]

SMALL ENOUGH

We need to compute

\[ \sum_{n=0}^{\infty} \frac{(12)^n}{n!} \]

... a digit approximation to \( e^{12} \)

Another approach:

Note \( e^{-x^2} = 1/e^{x^2} \)

\( e^{-x^2} \) is the sum of an alternating series
\[ e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \ldots \]

Since this alternating, we need to find a term \( a_n \) with \( |a_n| \leq 5 \times 10^{-10} \).

We see that \( a_8 = -6.3 \times 10^{-10} \)

so \[ \sum_{n=0}^{7} \frac{(-x^2)^n}{n!} \] is sufficiently close to \( e^{-x^2} \). It remains...
is the # we want

\[
\binom{8}{2} = \frac{2.56 \times 10^{-6}}{4.032 \times 10^{-4}} = 2 \times 10^{-1}
\]

\[
5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120
\]

\[
6! = 6 \cdot 120 = 720
\]

\[
7! = 7 \cdot 720 = 5040
\]

\[
8! = 8 \cdot 5040 = 40320
\]
Limit comparison test

Let \( A = \sum a_n \) and \( \sum b_n = K \)

If the series with \( a_n, b_n \geq 0 \)

No alternating series, please!

Suppose the behavior of \( A \) or \( B \)

and we can find

\[
L = \lim_{n \to \infty} \frac{a_n}{b_n}
\]
If $0 < L < \infty$, then the two 2 series behave the same way.

Example:  
\[ A = \sum \frac{\sqrt{n} + 25}{n + 7} \]

\[ B = \sum \frac{1}{\sqrt{n}} \]

$B$ diverges by the integral test.

\[ \frac{a_n}{b_n} = \frac{(\sqrt{n} + 25) / (n + 7)}{1 / \sqrt{n}} = \frac{n + 25\sqrt{n}}{n + 7} \]
$L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{n + 25\sqrt{n}}{n + 7} \right)$ Use Hopital's Rule

$= \lim_{n \to \infty} \left( \frac{1 + \frac{25}{2\sqrt{n}}}{1} \right)$

$= \lim_{n \to \infty} \left( 1 + \frac{25}{2\sqrt{n}} \right)$

$= 1$

The series behave the same