1. (20 points)

(a) (10 points) Find a partial fraction expansion for the function

\[
\frac{1}{x^3 - x^2 + 2x - 2}
\]

(b) (10 points) Calculate the integral

\[
\int \frac{dx}{x^3 - x^2 + 2x - 2}.
\]

Solution: (a) One notices that 1 is a root of the denominator. Polynomial division yields

\[
x^3 - x^2 + 2x - 2 = (x - 1)(x^2 + 2),
\]

so

\[
\frac{1}{x^3 - x^2 + 2x - 2} = \frac{1}{(x - 1)(x^2 + 2)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2}
\]

\[
= \frac{A(x^2 + 2) + (Bx + C)(x - 1)}{(x - 1)(x^2 + 2)} = \frac{Ax^2 + 2A + Bx^2 + Cx - Bx - C}{(x - 1)(x^2 + 2)} = \frac{(A + B)x^2 + (C - B)x + 2A - C}{(x - 1)(x^2 + 2)}
\]

By comparing numerators we must have \(A + B = 0\), \(C - B = 0\) and \(2A - C = 1\). From this we get \(A = 1/3\), \(B = -1/3\) and \(C = -1/3\).

(b) To calculate this integral, use the partial fraction expansion from (a). One gets:

\[
\int \frac{dx}{x^3 - x^2 + 2x - 2} = \int \left(\frac{1}{3(x - 1)} - \frac{x + 1}{3(x^2 + 2)}\right) dx
\]

\[
= \frac{1}{3} \left( \int \frac{dx}{x - 1} - \int \frac{x}{x^2 + 2} dx - \int \frac{1}{x^2 + 2} dx \right)
\]

\[
= \frac{1}{3} \left( \int \frac{dx}{x - 1} - \int \frac{x dx}{x^2 + 2} - \int \frac{dx}{x^2 + 2} \right).
\]
The first two integrals are done by substitution, \( u = x + 1 \) in the first integral and \( u = x^2 + 2 \) in the second integral. For the last summand we use the trigonometric substitution \( x = \sqrt{2} \tan(\theta) \) and hence \( dx = \sqrt{2} \sec^2(\theta) d\theta \). The last integral then becomes

\[
\int \frac{dx}{x^2 + 2} = \int \frac{\sqrt{2} \sec^2(\theta) d\theta}{2 \sec^2(\theta)} = \int \frac{d\theta}{\sqrt{2}} .
\]

Substituting back \( \theta = \arctan \left( \frac{x}{\sqrt{2}} \right) \) we get

\[
\int \frac{dx}{x^2 + 2} = \frac{1}{\sqrt{2}} \arctan \left( \frac{x}{\sqrt{2}} \right) .
\]

Combining the three calculations we get

\[
\int \frac{dx}{x^3 - x^2 + 2x - 2} = \frac{1}{3} \left( \ln |x + 1| + \frac{1}{2} \ln(x^2 + 2) + \frac{1}{\sqrt{2}} \arctan \left( \frac{x}{\sqrt{2}} \right) \right) + C .
\]

2. (20 points) Consider the curve \( y = x^{3/2} \)

(a) (10 points) Calculate the arc length function starting at \( x = 0 \).

2. (b) (10 points) Calculate the arc length from \( x = 4 \) to \( x = 8 \).

Solution: (a) \( y' = \frac{3}{2} \sqrt{x} \), so by substituting \( u = 1 + \frac{9}{4} x \) one gets

\[
s(t) = \int_0^t \sqrt{1 + \frac{9}{4} x} \, dx \\
= \frac{4}{9} \int_1^{1+9t/4} \sqrt{u} du \\
= \frac{8}{27} u^{3/2} \bigg|_1^{1+9t/4} \\
= \frac{8}{27} \left( 1 + \frac{9}{4} x \right)^{3/2} - \frac{8}{27}
\]

for \( t \geq 0 \).

(b) By the definition of the arc length function, \( s(4) \) is the arclength from \( t = 0 \) to \( t = 4 \).
and $s(8)$ is the arclength from $t = 0$ to $t = 8$, so the arclength from $t = 4$ to $t = 8$ is

$$s(8) - s(4) = \frac{8}{27} (19^{3/2} - 10^{3/2}) = \frac{8}{27} \left(19\sqrt{19} - 10\sqrt{10}\right).$$

3. **(20 points)** Consider region between the curve $y = \sin^2 x$ for $0 \leq x \leq \pi$ and the $x$-axis. (a) Find the volume of the solid of revolution about the $x$-axis.

3. (b) Find the volume of the solid of revolution about the $y$-axis. **Solution:** (a) This is a washer method problem. We have

$$V = \int_{0}^{\pi} \pi y^2 \, dx$$

$$= \pi \int_{0}^{\pi} \sin^4 x \, dx$$

$$= \pi \int_{0}^{\pi} \left(\frac{1 - \cos 2x}{2}\right)^2 \, dx$$

$$= \pi \int_{0}^{\pi} \frac{1}{4} (1 - 2 \cos 2x + \cos^2 2x) \, dx$$

$$= \pi \int_{0}^{\pi} \frac{1}{4} (1 - 2 \cos 2x + \frac{1 + \cos 4x}{2}) \, dx$$

$$= \frac{\pi}{8} \int_{0}^{\pi} (3 - 4 \cos 2x + \cos 4x) \, dx$$

$$= \frac{\pi}{8} \left(3x - 2 \sin 2x + \frac{\sin 4x}{4}\right)\bigg|_{0}^{\pi}$$

$$= \frac{3\pi^2}{8}.$$

(b) This is a shell method problem. We have

$$V = \int_{0}^{\pi} 2\pi xy \, dx$$

$$= 2\pi \int_{0}^{\pi} x \sin^2 x \, dx$$

$$= 2\pi \int_{0}^{\pi} x \left(\frac{1 - \cos 2x}{2}\right) \, dx$$

$$= \pi \int_{0}^{\pi} x \, dx - \pi \int_{0}^{\pi} x \cos 2x \, dx$$

$$= \frac{\pi x^2}{2} \bigg|_{0}^{\pi} - \pi \int_{0}^{\pi} x \cos 2x \, dx$$

$$= \frac{\pi^3}{2} - \pi \int_{0}^{\pi} x \cos 2x \, dx$$
The remaining integral requires integration by parts with

\[ u = x \quad dv = \cos 2x \, dx \]
\[ du = dx \quad v = \frac{\sin 2x}{2} \]

This gives

\[
\int_{0}^{\pi} x \cos 2x \, dx = \int_{x=0}^{x=\pi} u \, dv \\
= uv|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} v \, du \\
= \frac{x \sin 2x}{2}\big|_{x=0}^{x=\pi} - \int_{0}^{\pi} \frac{\sin 2x}{2} \, dx \\
= 0,
\]

so

\[
V = \frac{\pi^3}{2}.
\]

4. (20 points)

(a) (10 points) Use integration by parts to find a formula for

\[
\int x^n e^x \, dx \quad \text{in terms of} \quad \int x^{n-1} e^x \, dx
\]

(b) (10 points) Use this formula to find

\[
\int x^3 e^x \, dx.
\]

Solution: (a) Let \( u = x^n \) and \( dv = e^x \, dx \), so \( du = nx^{n-1} \) and \( v = e^x \). Then we have

\[
\int x^n e^x \, dx = \int u \, dv = uv - \int v \, du \\
= x^n e^x - n \int x^{n-1} e^x \, dx.
\]


(b) 

\[ \int x^3e^x \, dx = x^3e^x - 3 \int x^2e^x \, dx \]
\[ = x^3e^x - 3 \left( x^2e^x - 2 \int xe^x \, dx \right) \]
\[ = (x^3 - 3x^2)e^x + 6 \int xe^x \, dx \]
\[ = (x^3 - 3x^2)e^x + 6 \left( xe^x - \int e^x \, dx \right) \]
\[ = (x^3 - 3x^2 + 6x)e^x - 6 \int e^x \, dx \]
\[ = (x^3 - 3x^2 + 6x - 6)e^x + C \]

5. **(20 points)** Consider the integral 

\[ \int \frac{dx}{\sqrt{4x^2 - 12x}} \]

(a) (5 points) Write the quantity under the square root sign as a sum or difference of two squares.

(b) (5 points) Draw a right triangle in which one of the sides is the square root in the integer and another is a constant.

5. (c) (10 points) Evaluate 

\[ \int_3^4 \frac{dx}{\sqrt{4x^2 - 12x}}. \]

Solution: (a) \((2x - 3)^2 = 4x^2 - 12x + 9\) so \(4x^2 - 12x = (2x - 3)^2 - 3^2\).

(a) The triangle has hypotenuse \(2x - 3\), adjacent side 3 and opposite side \(\sqrt{4x^2 - 12x}\).

(c) We have

\[ \sqrt{4x^2 - 12x} = 3 \tan \theta \]
\[ 2x - 3 = 3 \sec \theta \]
\[ 2dx = 3 \sec \theta \tan \theta d\theta \]
so the indefinite integral is

\[
\int \frac{dx}{\sqrt{4x^2 - 12x}} = \frac{3}{2} \int \frac{\sec \theta \tan \theta}{3 \tan \theta} d\theta
\]

\[
= \frac{1}{2} \int \sec \theta d\theta
\]

\[
= \frac{1}{2} \log(\sec \theta + \tan \theta) + C
\]

\[
= \frac{1}{2} \log \left( \frac{2x - 3}{3} + \frac{\sqrt{4x^2 - 12x}}{3} \right) + C
\]

and

\[
\int_3^4 \frac{dx}{\sqrt{4x^2 - 12x}} = \left. \frac{1}{2} \log \left( \frac{2x - 3}{3} + \frac{\sqrt{4x^2 - 12x}}{3} \right) \right|_3^4
\]

\[
= \frac{1}{2} \left( \log \left( \frac{5}{3} + \frac{4}{3} \right) - \log \left( \frac{3 + 0}{3} \right) \right)
\]

\[
= \frac{\log(3)}{2}.
\]