1. **(20 points)** Evaluate the following integrals:

(a) (10 points)

\[ \int \frac{3x}{(x+1)(x^3+1)} \, dx. \]

(b) (10 points)

\[ \int_{0}^{\pi/2} \sin^4 x \, dx. \]

**Solution:** (a) By partial fractions we have

\[
\frac{3x}{(x+1)(x^3+1)} = \frac{3x}{(x+1)^2(x^2-x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-x+1} \tag{1}
\]

By comparing numerators we must have

\[ A + C = 0, \quad B + 2C + D = 0, \quad -B + C + 2D = 3 \]

and \[ A + B + D = 0. \]

From this we get \[ A = C = 0, \quad B = -1 \]

and \[ D = 1. \]

Alternatively, we can use Heaviside’s method to find the constants. Multiply both sides of (1) by \((x+1)^2\) and get

\[
\frac{3x}{x^2-x+1} = A(x+1) + B + (x+1)^2 \frac{Cx+D}{x^2-x+1}
\]

Setting \[ x = -1 \] gives

\[ B = \frac{-3}{3} = -1. \]
Subtracting the $B$ term from both sides of (1) gives

$$\frac{A}{x+1} + \frac{Cx + D}{x^2 - x + 1} = \frac{3x}{(x+1)(x^2+1)} + \frac{1}{(x+1)^2}$$

$$= \frac{3x + x^2 - x + 1}{(x+1)^2(x^2 - x + 1)}$$

$$= \frac{x^2 + 2x + 1}{(x+1)^2(x^2 - x + 1)}$$

$$= \frac{1}{x^2 - x + 1}.$$

From this we see that $A = 0$, $C = 0$ and $D = 1$ as before.

Thus one gets

$$\int \frac{3x}{(x+1)(x^3+1)}\,dx = - \int \frac{dx}{(x+1)^2} + \int \frac{dx}{x^2 - x + 1}.$$

The first integral is done by substitution $u = x+1$. For the second integral, we observe that

$$\frac{1}{x^2 - x + 1} = \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}}.$$

Therefore we use substitution $u = 2/\sqrt{3}(x - 1/2)$ and $du = 2/\sqrt{3}dx$, then the second integral is

$$\int \frac{dx}{x^2 - x + 1} = \frac{2}{\sqrt{3}} \int \frac{du}{u^2 + 1} = \frac{2}{\sqrt{3}} \tan^{-1} u = \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2}{\sqrt{3}} (x - \frac{1}{2}) \right).$$

Thus we have

$$\int \frac{3x}{(x+1)(x^3+1)}\,dx = \frac{1}{x + 1} + \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2}{\sqrt{3}} (x - \frac{1}{2}) \right) + K.$$

(b) We will use the double angle formulas $\sin^2 \theta = (1 - \cos 2\theta)/2$ and $\cos^2 \theta = (1 + \cos 2\theta)/2$. We have

$$\int_0^{\pi/2} \sin^4 x\,dx = \int_0^{\pi/2} \left( \frac{1 - \cos 2x}{2} \right)^2 \,dx$$

$$= \frac{1}{4} \int_0^{\pi/2} \left( 1 - 2 \cos 2x + \cos^2 2x \right) \,dx$$

$$= \frac{1}{4} \int_0^{\pi/2} \left( 1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) \,dx$$

$$= \frac{1}{4} \int_0^{\pi/2} \left( 3 - 4 \cos 2x + \cos 4x \right) \,dx$$

$$= \frac{1}{8} \left( 3x - 2 \sin 2x + \frac{\sin 4x}{4} \right) \bigg|_0^{\pi/2} = \frac{3\pi}{16}.$$
2. (20 points)

(a) (10 points)  Use integration by parts to find a formula for

\[ \int (\ln x)^n \, dx \quad \text{in terms of} \quad \int (\ln x)^{n-1} \, dx \]

(b) (10 points)  Use this formula to find

\[ \int (\ln x)^2 \, dx. \]

Solution:

(a) The integration by parts formula is

\[ \int u \, dv = uv - \int v \, du. \]

In this case we set

\[ u = (\ln x)^n \quad \text{and} \quad dv = dx \]
\[ du = \frac{n(\ln x)^{n-1} \, dx}{x} \quad \text{and} \quad v = x \]

so

\[ \int (\ln x)^n \, dx = \int u \, dv = uv - \int v \, du \]
\[ = x(\ln x)^n - \int x \frac{n(\ln x)^{n-1} \, dx}{x} \]
\[ = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx. \]

(b) Put \( n = 2 \) and \( n = 1 \) in the result of (a). Then we have

\[ \int (\ln x)^2 \, dx = x(\ln x)^2 - 2 \int \ln x \, dx, \]
\[ \int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C. \]

Combining these two equations, we have

\[ \int (\ln x)^2 \, dx = x(\ln x)^2 - 2x \ln x + 2x + C. \]
3. **(20 points)** (a) (10 points) Find the integral
\[
\int_{-1}^{0} \frac{dx}{\sqrt{x^2 + 4x + 3}}
\]

(b) (10 points) Find the integral
\[
\int_{4}^{6} \sqrt{8x - x^2} \, dx.
\]

**Solution:** (a) We have
\[
x^2 + 4x + 3 = (x + 2)^2 - 1
\]

We use the substitution \(x + 2 = \sec \theta\). Then we have
\[
\frac{dx}{\sqrt{x^2 + 4x + 3}} = \tan \theta
\]

so our integral is
\[
\int_{-1}^{0} \frac{dx}{\sqrt{x^2 + 4x + 3}} = \int_{0}^{\pi/3} \sec \theta \, d\theta
\]
\[
= \ln |\sec \theta + \tan \theta|_{0}^{\pi/3}
\]
\[
= \ln(\sqrt{3} + 2)
\]

(b) We have
\[
8x - x^2 = 16 - (x - 4)^2
\]

We use the substitution \(x - 4 = 4 \sin \theta\). From this we get
\[
\frac{dx}{\sqrt{8x - x^2}} = 4 \cos \theta \, d\theta
\]
so

\[ \int_4^6 \sqrt{8x - x^2} \, dx = \int_0^{\pi/6} (4 \cos \theta)4 \cos \theta \, d\theta \]

\[ = 16 \int_0^{\pi/6} \cos^2 \theta \, d\theta \]

\[ = 8 \int_0^{\pi/6} (1 + \cos 2\theta) \, d\theta \]

\[ = 8 \left( \theta + \frac{\sin 2\theta}{2} \right) \bigg|_0^{\pi/6} \]

\[ = \frac{4}{3} \pi + 2\sqrt{3}. \]
4. (20 points) Consider the curve 

\[ f(x) = 2x^{3/2} + 7 \]

(a) (10 points) Calculate the arc length function \( s(t) \) starting at \( x = 0 \), that computes the length of the curve from \((0, f(0))\) to \((t, f(t))\).

(b) (10 points) Calculate the arc length from \( x = 2 \) to \( x = 4 \).

**Solution:** (a) \( f'(x) = 3\sqrt{x} \), so the substitution \( u = 1 + 9x \) yields

\[
\begin{align*}
    s(t) &= \int_0^t \sqrt{1 + (3\sqrt{x})^2} \, dx \\
    &= \int_0^t \sqrt{1 + 9x} \, dx \\
    &= \frac{1}{9} \int_1^{1+9t} \sqrt{u} \, du \\
    &= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{1+9t} \\
    &= \frac{2}{27} (1 + 9t)^{3/2} - \frac{2}{27}
\end{align*}
\]

for \( t \geq 0 \).

(b) By the definition of the arc length function, \( s(4) \) is the arc length from \( t = 0 \) to \( t = 4 \) and \( s(2) \) is the arc length from \( t = 0 \) to \( t = 2 \), so the arc length from \( t = 2 \) to \( t = 4 \) is

\[
s(4) - s(2) = \frac{2}{27}(37)^{3/2} - \frac{2}{27} - \left( \frac{2}{27}(19)^{3/2} - \frac{2}{27} \right) = \frac{74}{27}\sqrt{37} - \frac{38}{27}\sqrt{19}
\]

5. (20 points) Consider region between the curves \( y = 2x \) and \( y = x^2 \).

(a) Find the volume of the solid of revolution about the \( x \)-axis.

(b) Find the volume of the solid of revolution about the \( y \)-axis.

**Solution:** (a) This is a washer method problem. The region bounded by the two graphs sits between \( x = 0 \) and \( x = 2 \) and in that interval \( 2x \) is the bigger function. We have
\[ V = \int_0^2 \pi [(2x)^2 - (x^2)^2] \, dx \\
= \pi \int_0^2 (4x^2 - x^4) \, dx \\
= \pi \left[ \frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 \\
= \pi \left( \frac{32}{3} - \frac{32}{5} \right) \\
= \frac{64\pi}{15} \]

(b) This is a shell method problem. We have

\[ V = \int_0^2 2\pi x (2x - x^2) \, dx \\
= 2\pi \int_0^2 (2x^2 - x^3) \, dx \\
= 2\pi \left[ \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 \\
= 2\pi \left( \frac{16}{3} - 4 \right) \\
= \frac{8\pi}{3} \]

OR:
We can integrate with respect to $y$ and use the washer method again:

\[
V = \int_0^4 \pi \left[ (\sqrt{y})^2 - \left( \frac{y}{2} \right)^2 \right] dy
\]

\[
= \pi \int_0^4 \left[ y - \left( \frac{y^2}{4} \right) \right] dy
\]

\[
= \pi \left[ \frac{y^2}{2} - \frac{y^3}{12} \right]_0^4
\]

\[
= \pi \left( 8 - \frac{16}{3} \right)
\]

\[
= \frac{8\pi}{3}
\]